Section 2.8

12. $p = 3$ and $q = 4$. Nul A is a subspace of $\mathbf{R}^3$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 3 entries, to match the columns of A. Col A is a subspace of $\mathbf{R}^4$ because each column vector has 4 entries.

14. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = (1/3)x_3$ and $x_2 = (-5/3)\,x_3$, with x_3 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any values of x_3 and x_4 (not both zero). For instance, set $x_3 = 3$ to obtain the vector $(1, -5, 3)$ in Nul A.

16. No. One vector is a multiple of the other, so they are linearly dependent and hence cannot be a basis for any subspace.

17. No. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A = \begin{bmatrix} 0 & 5 & 6 \\ 1 & -7 & 3 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & -10 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 23 \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbf{R}^3$ (as pointed out in Example 5).

18. Yes. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A = \begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & -8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & 0 & -5 \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbf{R}^3$ (as pointed out in Example 5).

20. No. The vectors are linearly dependent because there are more vectors in the set than entries in each vector. (Theorem 8 in Section 1.7.) So the vectors cannot be a basis for any subspace.

21. a. False. See the definition at the beginning of the section. The critical phrases "for each" are missing.
   b. True. See the paragraph before Example 4.
   c. False. See Theorem 12. The null space is a subspace of $\mathbf{R}^n$, not $\mathbf{R}^m$.
   d. True. See Example 5.
   e. True. See the first part of the solution of Example 8.

22. a. False. See the definition at the beginning of the section. The condition about the zero vector is only one of the conditions for a subspace.
   b. True. See Example 3.
   c. True. See Theorem 12.
   d. False. See the paragraph after Example 4.
   e. False. See the Warning that follows Theorem 13.

24. For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & -4 & 7 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ This corresponds to: } \begin{aligned} x_1 \quad - 4x_3 + 7x_4 &= 0 \\ x_2 + 5x_3 - 6x_4 &= 0. \\ 0 &= 0 \end{aligned}$$

Solve for the basic variables and write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 - 7x_4 \\ -5x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Notes: (1) A basis is a *set* of vectors. For simplicity, the answers here and in the text list the vectors without enclosing the list inside set brackets. This style is also easier for students. I am careful, however, to distinguish between a matrix and the set or list whose elements are the columns of the matrix.

(2) Recall from Chapter 1 that students are encouraged to use the augmented matrix when solving $A\mathbf{x} = \mathbf{0}$, to avoid the common error of misinterpreting the reduced echelon form of A as itself the augmented matrix for a nonhomogeneous system.

(3) Because the concept of a basis is just being introduced, I insist that my students write the parametric vector form of the solution of $A\mathbf{x} = \mathbf{0}$. They see how the basis vectors span the solution space and are obviously linearly independent. A shortcut, which some instructors might introduce later in the course, is only to solve for the basic variables and to produce each basis vector one at a time. Namely, set all free variables equal to zero except for one free variable, and set that variable equal to a suitable nonzero number.

28. The easiest construction is to write a 3×3 matrix in echelon form that has only 2 pivots, and let **b** be any vector in $\mathbf{R}^3$ whose third entry is nonzero.

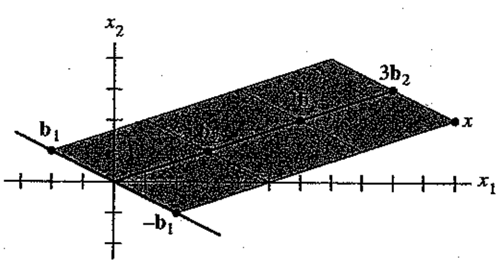
29. (Solution in *Study Guide*) A simple construction is to write any nonzero 3×3 matrix whose columns are obviously linearly dependent, and then make **b** a vector of weights from a linear dependence relation among the columns. For instance, if the first two columns of A are equal, then **b** could be $(1, -1, 0)$.

30. Since Col A is the set of all linear combinations of $\mathbf{a}_1, \ldots, \mathbf{a}_p$, the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ spans Col A. Because $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is also linearly independent, it is a basis for Col A. (There is no need to discuss pivot columns and Theorem 13, though a proof could be given using this information.)

Section 2.9

2. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $\mathbf{x}$ is formed from $\mathbf{b}_1$ and $\mathbf{b}_2$ using weights -1 and 3:

$$\mathbf{x} = (-1)\mathbf{b}_1 + 3\mathbf{b}_2 = (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$



4. As in Exercise 3, $[\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \mathbf{x}] = \begin{bmatrix} 1 & -3 & -7 \\ -3 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -7 \\ 0 & -4 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}$, and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

8. Fig. 2 suggests that $\mathbf{x} = 2\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{y} = 1.5\mathbf{b}_1 + \mathbf{b}_2$, and $\mathbf{z} = -\mathbf{b}_1 - .5\mathbf{b}_2$. If so, then $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$, and $[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -.5 \end{bmatrix}$. To confirm $[\mathbf{y}]_{\mathcal{B}}$ and $[\mathbf{z}]_{\mathcal{B}}$, compute

$$1.5\mathbf{b}_1 + \mathbf{b}_2 = 1.5\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \mathbf{y} \text{ and } -\mathbf{b}_1 - .5\mathbf{b}_2 = -1\begin{bmatrix} 0 \\ 2 \end{bmatrix} - .5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix} = \mathbf{z}.$$

10. The information $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$. For Nul A,

$$[A \ \ \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{aligned} x_1 + 3x_3 &= 0 \\ x_2 - 3x_3 - 7x_5 &= 0 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

x_3 and x_5 are free variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

From this, dim Col $A = 3$ and dim Nul $A = 2$.

14. The five vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 of the original matrix form a basis for H, so dim $H = 2$.

20. A 4×5 matrix A has 5 columns. By the Rank Theorem, rank $A = 5 - \dim \text{Nul } A$. Since the null space is three-dimensional, rank $A = 2$.