1.8 #6

We seek $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}.$$ ✓

We solve the linear system with augmented matrix $\begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{bmatrix}$. ✓

Reducing that matrix to rref we obtain

$$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ✓

✓ Since the rref for the linear system has no pivot in the last column, the system is consistent, but there will be a free variable, x_3, so the solution is not unique.

✓ In parametric vector form, the pre-images of $\begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$ look like $\bar{x} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ for $t \in \mathbb{R}$.

One such preimage is $\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$. ✓

1.8 #12

$\bar{b}$ will be in the range $\bar{x} \to A\bar{x}$ provided we can find $\bar{x} \in R^4$ such that $A\bar{x} = \bar{b}$. ✓ We consider the augmented matrix

✓ $$\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \overset{rref}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ ✓

✓ Since the rref of $[A|\bar{b}]$ has a pivot in the last column, no solution exists for $A\bar{x} = \bar{b}$.

Thus, $\bar{b}$ is not in the range of $\bar{x} \to A\bar{x}$ ✓

1.8 #28

Our author informs us that for any vectors $\bar{x}$ and $\bar{y}$ in $\mathbb{R}^n$ the set of vectors in the parallelogram determined by $\bar{x}$ and $\bar{y}$ has the form $a\bar{x} + b\bar{y}$ for $0 \le a \le 1$ and $0 \le b \le 1$.

Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\bar{z}$ is a point in P.

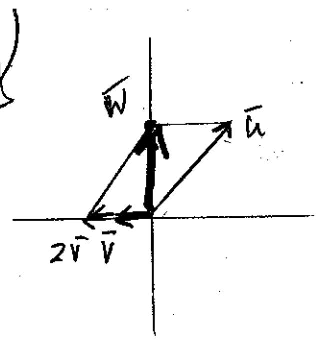
$\bar{z} = a\bar{u} + b\bar{v}$ for some $0 \le a \le 1$ and $0 \le b \le 1$.

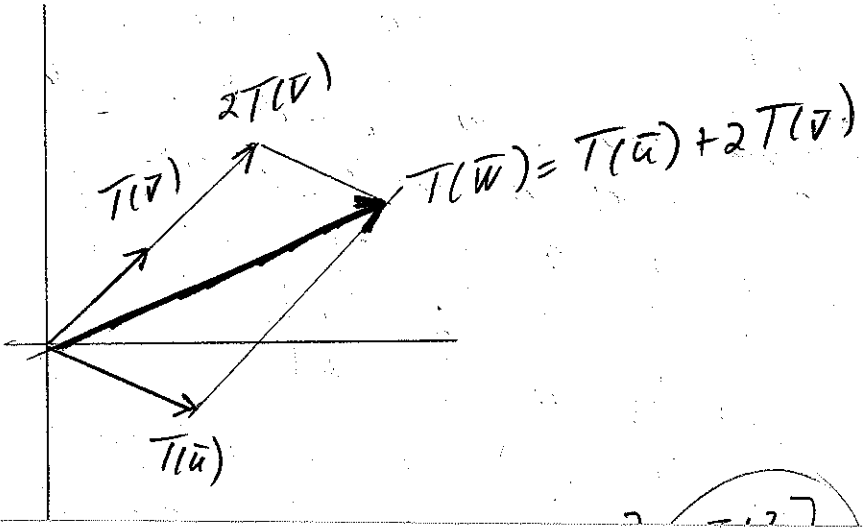
$$
\begin{aligned}
T(\bar{z}) &= T(a\bar{u} + b\bar{v}) \\
&= T(a\bar{u}) + T(b\bar{v}) \\
&= aT(\bar{u}) + bT(\bar{v})
\end{aligned}
$$

So, $T(\bar{z})$ lies in the parallelogram determined by $T(\bar{u})$ and $T(\bar{v})$.

Hence, the image of any point in P under the linear transformation T is in the parallelogram determined by $T(\bar{u})$ and $T(\bar{v})$.

1.8 #18





$\bar{W} = \bar{u} + 2\bar{v}$

$$
\begin{aligned}
T(\bar{W}) &= T(\bar{u} + 2\bar{v}) \\
&= T(\bar{u}) + 2T(\bar{v})
\end{aligned}
$$

1.9 #26

Here we consider the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\bar{x}) = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix} \bar{x}$.

Take $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ can we find $\bar{x} \in \mathbb{R}^3$ such that $T(\bar{x}) = \bar{b}$?

Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & b_1 \\ 3 & -7 & 4 & b_2 \end{array}\right] \sim \begin{bmatrix} 1 & 4 & -5 & b_1 \\ 0 & -19 & 19 & -3b_1 + b_2 \end{bmatrix}$$

There is a pivot in each row of the matrix of coefficients, so T is <u>onto</u> $\mathbb{R}^2$. However, there is no pivot in column three, so T is not 1-1.

✓

1.9 #28

Here we have $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\bar{x}) = [\bar{a}_1 \ \bar{a}_2]\bar{x}$ where the columns are linearly independent. So, by Theorem 12, T is 1-1 and onto. ~~We get the onto~~ part because $\{\bar{a}_1, \bar{a}_2\}$ spans $\mathbb{R}^2$.

1.8 #30

Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\bar{x}) = A\bar{x} + \bar{b}$ where A is $m \times n$ and $\bar{b} \in \mathbb{R}^m$ and $\bar{b} \neq \bar{0}$.

Consider $\bar{x}, \bar{y} \in \mathbb{R}^n$.

$T(\bar{x}) = A\bar{x} + \bar{b}$

$T(\bar{y}) = A\bar{y} + \bar{b}$

So, $T(\bar{x}) + T(\bar{y}) = A(\bar{x} + \bar{y}) + 2\bar{b}$

But, $T(\bar{x} + \bar{y}) = A(\bar{x} + \bar{y}) + \bar{b}$

and since $\bar{b} \neq \bar{0}$ $2\bar{b} \neq \bar{b}$.

So, $T(\bar{x}) + T(\bar{y}) \neq T(\bar{x} + \bar{y})$

✓ Hence, T is not a linear transformation.

1.8 #32

Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$.

Is T linear?

Let $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c = -2$

$T(-2\bar{x}) = T\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$-2\,T(\bar{x}) = -2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$

So, $T(-2\bar{x}) \neq -2\,T(\bar{x})$

✓ T is not linear because scalar multiplication is not preserved.