10. a) For what values of h is $\bar{V}_3$ in Span $\{\bar{V}_1, \bar{V}_2\}$?

$\bar{V}_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$, $\bar{V}_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}$, $\bar{V}_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$

To show $\bar{V}_3$ is in the Span $\{\bar{V}_1, \bar{V}_2\}$, we must show that $\bar{V}_1$ and $\bar{V}_2$ are a linear combination of $\bar{V}_3$.

write the vectors $\bar{V}_1, \bar{V}_2$ and $\bar{V}_3$ in an augmented matrix $\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix}$ now reduce to rref using row operations

$5R_1 + R_2 \to R_2$ $\quad 3R_1 + R_3 \to R_3$

$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -3 & 6 & h \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 6+h \end{bmatrix}$

this corresponds to the system
$x_1 - 2x_2 = 2$
$0 = 1$
$0 = 6+h$

Equation 3 cannot be solved since $0 \neq 1$ and therefore the system is inconsistent and no values of h would put $\bar{V}_3$ in Span$\{\bar{V}_1, \bar{V}_2\}$

b) For what values of h is $\{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$ linearly dependent?

To find the values of h that make $\{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$ linearly dependent, put $\bar{V}_1, \bar{V}_2$ and $\bar{V}_3$ in an augmented matrix and reduce to reduced row echelon form.

$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \xrightarrow[3R_1 + R_3 \to R_3]{5R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6+h & 0 \end{bmatrix} \xrightarrow[(-6-h)R_2 + R_3 \to R]{-2R_2 + R_1 \to R_1} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This corresponds to the system
$x_1 - 2x_2 = 0$
$x_3 = 0$
x_2 is free

Since x_2 is free there are infinitely many solutions to the system, which makes $\bar{V}_1, \bar{V}_2$, and $\bar{V}_3$ linearly dependent.

Any value of h would make $\{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$ linearly dependent because the pivot in the second row can be multiplied by any number to cancel out any value of $h+6$. Therefore the last row will always contain all zeros and since x_2 is a free variable the vectors are linearly dependent.