1.4 12, 18, 20, 40

12.) $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

The augmented matrix for $ax = b$ is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \underset{\sim}{3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \underset{\sim}{\begin{matrix} -R_2 + R_3 \to R_3 \\ \frac{1}{5}R_2 \to R_2 \end{matrix}}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & -2 & -2 \end{bmatrix} \underset{\sim}{-\frac{1}{2}R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{\sim}{\begin{matrix} -R_3 + R_2 \to R_2 \\ -R_3 + R_1 \to R_1 \end{matrix}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{-2R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The unique solution to the linear system is $(3/5, -4/5, 1)$.

Written as a vector, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the solution is

$$\bar{X} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}. \checkmark$$

18. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\bar{x}=\bar{y}$ have a solution for each $\bar{y}$ in $\mathbb{R}^4$?
Theorem 4 says that if there is a pivot in each row of the coefficient matrix then the columns of the matrix span $\mathbb{R}^n$ and each equation $B\bar{x}=\bar{y}$ would have a solution.
To find pivots, reduce the matrix to row reduced echelon form

$$B=\begin{bmatrix}1&3&-2&2\\0&1&1&-5\\1&2&-3&-7\\-2&-8&2&-1\end{bmatrix}\xrightarrow{-1R_1+R_3\to R_3}\begin{bmatrix}1&3&-2&2\\0&1&1&-5\\0&-1&-1&5\\-2&-8&2&-1\end{bmatrix}\xrightarrow{2R_1+R_4\to R_4}\begin{bmatrix}1&3&-2&2\\0&1&1&-5\\0&-1&-1&5\\0&-2&-2&3\end{bmatrix}\xrightarrow{-3R_2+R_1\to R_1}\begin{bmatrix}1&0&-5&17\\0&1&1&-5\\0&-1&-1&5\\0&-2&-2&3\end{bmatrix}$$

$$\xrightarrow{1R_2+R_3\to R_3}\begin{bmatrix}1&0&-5&17\\0&1&1&-5\\0&0&0&0\\0&-2&-2&3\end{bmatrix}\xrightarrow{2R_2+R_4\to R_4}\begin{bmatrix}1&0&-5&17\\0&1&1&-5\\0&0&0&0\\0&0&0&-7\end{bmatrix}\xrightarrow{-1/7R_4\to R_4}\begin{bmatrix}1&0&-5&17\\0&1&1&-5\\0&0&0&0\\0&0&0&1\end{bmatrix}\xrightarrow{5R_4+R_2\to R_2}\begin{bmatrix}1&0&-5&17\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow{-17R_4+R_1\to R_1}\begin{array}{r}\\ \\ \text{no pivot}\to\\ \\ \end{array}\begin{bmatrix}1&0&-5&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

row reduced echelon form

There is not a pivot in every row because there is no pivot in row three.
Therefore, according to Theorem 4, the columns of B do not span $\mathbb{R}^4$ and the equation $B\bar{x}=\bar{y}$ does not have a solution for each $\bar{y}$ in $\mathbb{R}^4$.

20. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B above? Do the columns of B span $\mathbb{R}^3$.
The row reduced echelon form of matrix B is

$$B=\begin{bmatrix}1&0&-5&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

as found in problem 18. According to Theorem 4 if there is not a pivot in each row, then every vector in $\mathbb{R}^4$ cannot be written as a linear combination of the columns of Matrix B. This is because the third row contains all zeros and some values in a vector would make the system inconsistent. Where $0=b$ where b is nonzero would be in the reduced echelon matrix.
The columns of B do span $\mathbb{R}^3$ because the third row could be switched with the fourth by a row operation. Then the first three rows of the matrix would have a pivot and B would span $\mathbb{R}^3$ according to Theorem Four.

40. Determine if the columns of the matrix span $\mathbb{R}^4$.

$$\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}$$

Theorem Four states that if each row has a pivot then the columns of the matrix will span $\mathbb{R}^4$.

To determine whether or not each row has a pivot reduce the matrix to the row reduced echelon form. Each row should have one "1" that has only zeros above and below it. Also each "1" should be farther to the right than the "1" in the previous column.

By using the row operation tool on the computer, the matrix was reduced to

$$\begin{bmatrix} 1 & 0 & -\frac{7}{13} & 0 & 0 \\ 0 & 1 & -\frac{2}{13} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Each row does have a pivot so the columns of the matrix do span $\mathbb{R}^4$ according to Theorem Four.