Assignment #6

2.2 #6

$$\begin{cases} 8x_1 + 5x_2 = -9 \\ -7x_1 + -5x_2 = 11 \end{cases}$$

Let $A = \begin{pmatrix} 8 & 5 \\ -7 & -5 \end{pmatrix}$

By theorem 4, $A^{-1} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix}$

In matrix form

$$\begin{pmatrix} 8 & 5 \\ -7 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{pmatrix}\begin{pmatrix} 8 & 5 \\ -7 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{pmatrix}\begin{pmatrix} -9 \\ 11 \end{pmatrix} =$$

Our solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

2.2 #8 ✓

Suppose A is invertible and $AD = I$.

We multiply both sides of $AD = I$ by A^{-1}

$$A^{-1}(AD) = A^{-1}I = A^{-1}$$

$$(A^{-1}A)D = A^{-1}$$

$$ID = A^{-1}$$

$$D = A^{-1}$$

So A invertible and $AD = I \implies D = A^{-1}$

2.2 #12) Suppose $A_{n\times n}$ is invertible and for $B_{n\times p}$ then $A^{-1}B$ can be computed by row reduction.

Proof Suppose $A_{n\times n}$ is invertible and $B_{n\times p}$ is another matrix. Since A is invertible, $A \sim I_n$ and there exis elementary matrices $E_1, E_2, \ldots, E_p$ such that $E_p \cdots E_1 A = I_n$. It follows that $(E_p \cdots E_1) = A^{-1}$. It also follows that applying the same sequence of elementary row operations to B yields $(E_p \cdots E_1)B = A^{-1}B$.

So, if $[A \ \ B] \sim [I \ \ X]$ then $X = A^{-1}B$.

2.2 #13) Suppose $AB = AC$ where B and C are $n\times p$ and A is invertible.

We ~~now~~ have $A^{-1}(AB) = A^{-1}(AC)$

$$(A^{-1}A)B = (A^{-1}A)C$$
$$IB = IC$$
$$B = C$$

So, $AB = AC$ and A invertible $\Rightarrow B = C$.

This does not follow if A is not invertible as we discovered in 2.1/#10.

2.2 #16

Suppose A and B are $n \times n$ and B is invertible and AB is invertible.

Consider the product $(AB)B^{-1}$. Of course $(AB)B^{-1} = A(BB^{-1}) = AI = A$.

By the highlighted remark on p. 122

$[(AB)B^{-1}]$ is invertible and $[(AB)B^{-1}]^{-1} = B(AB)^{-1}$.

Now consider

$$A[B(AB)^{-1}] = (AB)(AB)^{-1} = I.$$

So, A^{-1} exists and $A^{-1} = B(AB)^{-1}$.

Hence A, B $n \times n$ and B and AB invertible $\Rightarrow$ A invertible

2.2 #18

Suppose P is invertible and $A = PBP^{-1}$.

We solve for B as follows

$$P^{-1}(PBP^{-1})P = P^{-1}AP$$

$$(P^{-1}P)B(P^{-1}P) =$$

$$IBI =$$

So, $\boxed{B = P^{-1}AP}$

2.2 #22 Suppose $A = [\bar{a}_1 \cdots \bar{a}_n]$ is invertible.

By Thm 5 $A\bar{x} = \bar{b}$ is consistent for each $\bar{b} \in \mathbb{R}^n$.

By Thm 4 in Section 1.4 the columns of A span $\mathbb{R}^n$.

So, $A = [\bar{a}_1 \cdots \bar{a}_n]$ invertible $\Rightarrow \{\bar{a}_1, \ldots, \bar{a}_n\}$ spans $\mathbb{R}^n$.

2.2 #32

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\overset{\substack{-4R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3}}{}} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{-2R_2 + R_3 \to R_3} \left[\begin{array}{cccccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$$

So, $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}^{-1}$ does not exist because it does not have a pivot in each row.

2.2 #34

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 2 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots -\frac{1}{n} & \frac{1}{n} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 2 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots -\frac{1}{n} & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$