MATH 306 Linear Algebra
Test #3

Name

Fall 2009

I. Definitions (4 points each). Write definitions of the specified terms. Be sure to write complete English sentences in stating the definitions.

1. Define what it means for a mapping T: U → V to be a *linear transformation*. You may use the word "preserved" in your definition.

✓ A mapping $T: U \to V$ is a linear transformation provided that addition and scalar multiplication are preserved under T

OR ✓ A mapping $T: U \to V$ is a linear transformation provided that for each $\bar{x}, \bar{q} \in U$ and $k \in R$ we have $T(\bar{x}+\bar{q}) = T(\bar{x}) + T(\bar{q})$ and $T(k\bar{x}) = k\,T(\bar{x})$.

2. Define what it means to say a set H is a subspace of a vector space V.

A set H is a subspace of a vector space V provided $H \subseteq V$ and $\bar{0}_V \in H$ and H is closed under addition and scalar multiplication.

II Exercises (7 points each).

3. P_3 is the vector space of polynomials of degree three or less.
Is the set S = {**p** : **p**(t) = a + bt^2 for some scalars a and b} a subspace of P_3? Justify your answer.

Yes S is a subspace of P_3 as we show below.

✓ Clearly $S \subseteq P_3$

✓ The zero of P_3 is $\bar{z}$ where $\bar{z}(t) = 0 + 0t + 0t^2$ and that zero vector is in S.

✓ Take $\bar{p}, \bar{q} \in S$ and $k \in R$

✓ $\bar{p}(t) = a_1 + b_1 t^2$, $\bar{q}(t) = a_2 + b_2 t^2$ for some $a_1, b_1, a_2, b_2 \in R$.

$(\bar{p}+\bar{q})(t) = \bar{p}(t) + \bar{q}(t) = (a_1 + a_2) + (b_1 + b_2)t^2$

✓ So, S is closed under addition

✓ $(k\bar{p})(t) = k\,p(t) = (ka_1) + (kb_1)t^2$

✓ So, S is closed under scalar multiplication

4. Suppose we define the 3 x 4 matrix A as below. Complete the following items.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

✓ a. A basis for Col A = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

✓✓ b. A basis for Nul A = $\left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

✓ c. A basis for Row A = $\{ [1, -2, 0, 1], [0, 1, 2, -1], [0, 0, 0, 1] \}$

✓ d. Rank A = 3

✓ e. Rank A + dim Nul A = 4

✓ f. Rank A^T = 3

5. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$. Show the steps one uses in finding all the eigenvalues of A. Specify those eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{bmatrix} = (2-\lambda)\left[(1-\lambda)(3-\lambda) + 1\right]$$

$$= (2-\lambda)\left[3 - 4\lambda + \lambda^2 + 1\right] = (2-\lambda)(2-\lambda)(2-\lambda)$$

So, the only eigenvalue is 2. ✓✓

6. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

a. Specify bases for each of the following:

A basis for Col $A^T = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ✓

A basis for Nul A = $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ ✓

b. Show that each vector in Nul A is orthogonal to each vector in Col A^T.

✓ Take $\bar{x} = k_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ in Nul A and $\bar{y} = k_2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in Col A^T

Consider

$$\bar{x} \cdot \bar{y} = \left(k_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) \cdot \left(k_2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \checkmark$$

$$= (k_1 k_2) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + (k_1 k_3) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

$$= 0 + 0$$

$$= 0 \checkmark$$

So, $\bar{x}$ and $\bar{y}$ are orthogonal vectors ✓

7. $B = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$, are both bases for R^2.

Find the change-of-coordinates matrix from B to C, and show how you find that matrix. Also, show how to use that matrix to obtain $[\mathbf{v}]_C$ if $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ -1 & 2 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$[\bar{V}]_C = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$. An eigenvalue of A is 3. Show how to find a basis for the corresponding eigenspace. Specify the basis you find.

Solve $A\bar{x} = 3\bar{x}$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \bar{x} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

So, a basis for the eigenspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$