

## Fibonacci Primes

The Fibonacci numbers are a famous sequence of numbers, generally attributed to Leonardo of Pisa, which are given by the following recurrence

$$F_{n+1} = F_n + F_{n-1} \quad (n \geq 1; F_0 = 0, F_1 = 1).$$

While it is true that the Fibonacci numbers can be found in Pascal's triangle (sum the binomial coefficients on the back diagonals), the main purpose of this lab is to explore the Fibonacci numbers which are prime.

It is not known whether or not there are infinitely many Fibonacci primes. The first ten Fibonacci primes are

$$F_3, F_4, F_5, F_7, F_{11}, F_{13}, F_{17}, F_{23}, F_{29}, \text{ and } F_{43}.$$

Except for  $F_4$ , it is true that if  $F_n$  is prime, then  $n$  is prime. The converse is not true however. For example,  $F_{19} = 4181 = 37 \times 113$ . To date, the largest known Fibonacci prime is  $F_{81839}$ . (For an up-to-date listing of Fibonacci primes, check out Neil Sloane's *On-Line Encyclopedia of Integer Sequences*.)

### Exercises

1. Prove that if  $1 \leq k \leq n - 1$ , then  $\binom{n}{k} \equiv 0 \pmod{n}$  if and only if  $n$  is prime. (*Hint:* To establish the necessary condition, prove its contrapositive. That is, suppose that  $n$  is composite, let  $p$  be its largest prime divisor, and then show that the binomial coefficient  $\binom{n}{p}$  is not equal to 0 modulo  $n$ .)
2. Find the first six Fibonacci primes using the Fibonacci recurrence. That is,  $F_3 = ?$ ,  $F_4 = ?$ , etc.
3. Use exercise 1 and the PascGalois JE software to show that  $F_{13}$  is prime.
4. Is  $F_2$  prime? Use exercise 1 and the PascGalois JE software to show that  $F_{19}$  is not prime.
5. Use induction to prove that

$$\sum_{k=0}^n \binom{k}{n-k} = F_{n+1}.$$

6. Use induction to prove that if  $n \geq 1$ , then  $\gcd(F_n, F_{n-1}) = 1$ . (*Hint:*  $\gcd(a, b) = \gcd(b, a - b)$ .)