The Lucas Correspondence Theorem

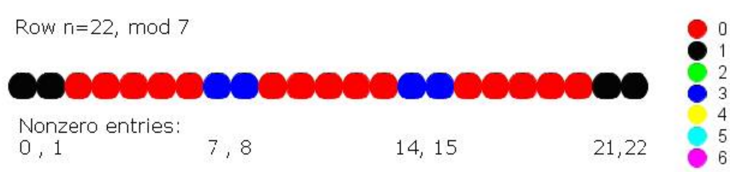
Introduction

Given a prime number p and integers r, k such that $r > 0$ and $0 \leq k \leq r$, we will write the base p representations of r and k as follows:

$$
\begin{aligned}
r &= r_m p^m + \cdots + r_1 p + r_0 \ (0 \leq r_i < p), \\
k &= k_m p^m + \cdots + k_1 p + k_0 \ (0 \leq k_i < p),
\end{aligned}
$$

where m is chosen such that $p^m \leq r < p^{m+1}$.

We will consider row r of Pascal's Triangle modulo p. As an example, let $p = 7$ and $r = 22$; that is, we will consider the 22^{nd} row of Pascal's Triangle modulo 7:



If we write the row number and the locations of the non-zero entries, all in base 7 (including leading zeros, if necessary), we get the following table:

n=22	3	1	Row number (base 7)
k=0	0	0	Nonzero locations (also base 7)
1	0	1	
7	1	0	
8	1	1	
14	2	0	
15	2	1	
21	3	0	
22	3	1	

From examples such as this one, we wish to infer a method of predicting the nonzero entries of row n of Pascal's triangle modulo p.

Activity Instructions

1. Following the above example, create similar tables for rows 14, 16, 18 and 20 of Pascal's triangle modulo 7. In each case, what is the relationship between the base p digits of the row number and the base p digits of the locations of the nonzero entries?

2. For a few different values of k from row $r = 18$ of the (mod 7) triangle, calculate the binomial coefficient $\binom{r_j}{k_j}$ for each j. In each case, compare your results to the value of $\binom{r}{k}$. Write out your results.

3. For each of the examples you considered in # 2, look for a connection between the values of $\binom{r_j}{k_j}$ mod 7 and the value of $\binom{r}{k}$ mod 7. In particular, consider the product $\prod_j \binom{r_j}{k_j}$ mod 7. Keep in mind that when $a < b$, $\binom{a}{b} = 0$.

   For example: If $r = 18 = 24_7$ and $k = 10 = 13_7$, then:

$$\begin{aligned}
\binom{r}{k} &= \binom{18}{10} = \frac{18!}{10!8!} = 43758 \equiv 1 \mod 7, \\
\binom{r_0}{k_0} &= \binom{4}{3} = 4, \\
\binom{r_1}{k_1} &= \binom{2}{1} = 2, \quad \text{and} \\
\prod_j \binom{r_j}{k_j} &= \binom{r_0}{k_0}\binom{r_1}{k_1} = 4 \cdot 2 = 8 \equiv 1 \mod 7.
\end{aligned}$$

   Come up with a conjecture about the connection between $\binom{r}{k}$ mod 7 and $\prod_j \binom{r_j}{k_j}$ mod 7.

4. Test your conjecture for some other prime moduli p and rows r. (So far we've only really considered $p = 7$ and $r = 18$.) Consider at least two other values of p; for each of these, repeat the procedures described in #1 and #2 above for at least two or three different rows of Pascal's triangle modulo p. Does your conjecture from # 3 seem to hold up? Do you think it is true in general?

5. Look up the "Lucas Correspondence Theorem" on the internet. (For example, you can find it on mathworld.wolfram.com.) Compare what you find to what you came up with in #3–4 above.

6. Describe how the Lucas Correspondence Theorem reveals the locations of the zero versus non-zero entries of Pascal's triangle modulo a prime p.