

# An Introduction to Isomorphism

## Groups of Order 4

In this lab, we are going to investigate isomorphisms of group structures. Recall that two group operations are isomorphic if there is a bijection between the sets and this bijection preserves the operation. That is,

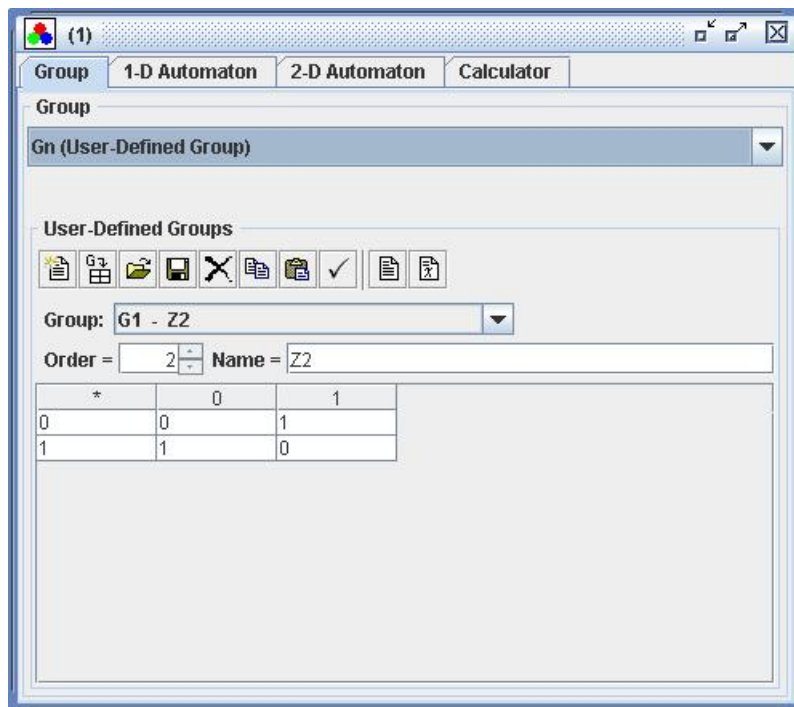
$$\phi(a * b) = \phi(a) *' \phi(b)$$

for all  $a$  and  $b$  in the domain. Also recall that we can visually tell if two group structures are isomorphic if we look at their operation tables. We must be able to match the middle of one table with the middle of the other, with possibly a rearrangement of the rows and columns.

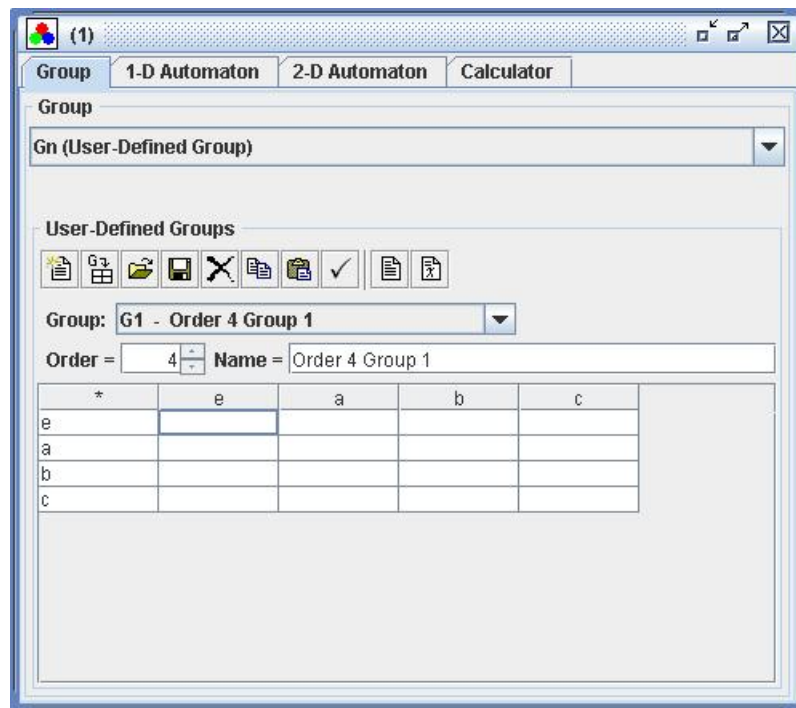
It is an easy exercise to show that there is only one group of order 2 and one group of order 3 (up to isomorphism). When we construct operation tables for four element groups we find that there are cells in the table that we have the ability to choose several different elements. Each choice we make can possibly lead to a valid group operation. Some of these look the same and others look different. We will use the PascGalois JE program to examine their structure.

### Exercises:

1. What are the rules you need to follow when constructing a group operation table? That is, what must be true about each column? What must be true about each row? How do the group properties translate to table rules? . . .
2. By hand, create the operation tables for a group operation on a set of four elements  $\{e, a, b, c\}$  where we will use  $e$  as the group identity. Do not eliminate a possible table even if it looks like one of the others you created. How many tables did you get? List them here.
3. Load each of the group tables from the above exercise into the PascGalois JE program as follows.
  - (a) Open up a new internal window for each of your possibly different groups of order 4, and tile the windows so you can see all of them at the same time.
  - (b) In the Group tab select Gn User-Defined Groups in the type.



- (c) In the User-Defined Groups table, increase the number of elements to 4.
- (d) Now fill in the table. Put the group elements in the far left column and note that they are automatically copied to the appropriate column headers at the top. Now fill in the body of the table, you can use either tab or enter to move from cell to cell. Tab will move horizontally and enter will move vertically. Also, since this group is not going to be  $Z_2$  change the name of the group to something more representative.



- (e) Now we will graph the structure. Click on the 1-D Automaton tab.
- (f) Under the options tab you should see an  $e$  in the Default Element box. That is because the PascGalois JE program will automatically find the identity element of a structure and place it here. If you do not see an  $e$  in this spot click on the Use Group Identity button. If you get an error message then you know that you do not have a unique identity element in your structure and hence it is not a group. Check your input and then your work you did in constructing the table.
- (g) Since this is not  $Z_n$  we will also need to change the seed. Click on the Seed tab and change the number of columns to 2. Now in the grid you should have one row and two columns. In the first column put in an  $a$  and in the second column put in a  $b$ .
- (h) Click on the Image tab and graph the image.
- (i) Also try  $a$  and  $c$  as seed elements and then  $b$  and  $c$  as seed elements. Use the pair that seems to give you more triangles (structure) in the image.
- (j) Do this for all of the tables you made, remember, one per window.
4. Looking at all of the images side by side which ones look the same? Remember this course focuses on structure not elements. If the colors are a bit different that is just a difference in elements but if the triangle placements are different then this is a difference in structure.
5. How many different structures are there? Which ones are the same and which are different?
6. For the structures that are the same create an isomorphism between them. With the PascGalois JE program you can drag-and-drop colors from one window to another. So if you put your mouse over

an element in the color correspondence of one window, click and drag it to an element in the color correspondence of another window and release the mouse button the color from the first element will be dropped into the color of the second element. Now construct the necessary isomorphisms, once you have what you believe to be an isomorphic map, drag-and-drop the colors from one window to another using this map and regraph the image. How do the two images compare now?

7. Do any of the structures look like  $Z_4$ ? If so, which one(s)?
8. Looking at the results of your work above, how many “different” groups of order 4 are there?