

Quotient Groups of \mathbb{Z}_n

One of the most important results often covered in a first semester undergraduate algebra course is the Fundamental Homomorphism Theorem which establishes the relationship between homomorphic images and quotient groups. To understand this result, you must be comfortable with both homomorphic images and quotient groups. However, quotient groups can often be a difficult concept to understand. Building groups out of cosets (which are equivalence classes defined using a normal subgroup) can be unsettling when first studied. The purpose of this project is to visualize the coset identifications created when constructing a quotient group. Hopefully seeing graphical representations of these identifications will make it easier for you to see these equivalence classes as points in a quotient group. Even though each coset has internal structure (recall that cosets are subsets that partition the group), it is treated as a point in the quotient group.

Exercises:

1. Construct Pascal's triangle mod 10 with 64 rows, 125 rows, and 128 rows. Study the patterns you see very carefully. Notice that there seem to be competing themes within the triangle. That is, there appear to be two distinct patterns, one superimposed on top of the other. Based on the colors you see and the group elements to which they correspond, can you predict what may be causing this visual effect?

Note that $\{0, 5\}$ is a subgroup of \mathbb{Z}_{10} . Since every subgroup of an abelian group is normal, we know that coset addition is well-defined for $\mathbb{Z}_{10}/\langle\{0, 5\}\rangle$ and, hence, it is a group. Certainly $\mathbb{Z}_{10}/\langle\{0, 5\}\rangle \cong \mathbb{Z}_5$. The cosets are $\{0, 5\}$, $\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, and $\{4, 9\}$. Which coset acts as the identity for the quotient group? Which coset(s) generate the quotient group?

2. Go back to your computer drawing of Pascal's triangle mod 10. We will recolor the elements by their cosets, that is, we will color all the elements of a coset the same color and different cosets will get different colors. This is the color equivalent to viewing each coset as a single element so we will see an image of the quotient group $\mathbb{Z}_{10}/\langle\{0, 5\}\rangle$. For the first coset $\{0, 5\}$, click on the 0 element in the color correspondence to the right. Hold down the control key and click the element 5. At this point both 0 and 5 should be highlighted. Now select Colors \succ Group Elements. At this point both 0 and 5 should be the same color. The program actually did more than that, it grouped 0 and 5 together so now if you double click on 0 or 5 and change its color both colors will change. The same is true for making the colors transparent. Now group 1 and 6 together, 2 and 7 and so on. After you are finished you should have each coset a different color and all the elements of a coset are the same color. Again, this is a coloring of $\mathbb{Z}_{10}/\langle\{0, 5\}\rangle$. Redraw the triangle with these color identifications. What do you see? Can you explain the observed pattern based on group structure and quotient groups? Draw the Fundamental Homomorphism Theorem commutative diagram that represents this investigation.

You probably found the process of identifying the colors of the cosets to be a little tedious, even for a relatively small group like \mathbb{Z}_{10} . To make coset identifications easier we built in two options under the Colors menu. There are left coset and a right coset options which will automatically identify the left cosets (or right cosets). As you know, if your subgroup is normal these are the same but since the PascGalois JE program will allow you to do identifications using non-normal subgroups we needed to put in both options. To test this select Colors \succ Reset to Default Color Scheme to reset and ungroup the elements. Select the element 5 from the color correspondence, and just the 5. Now select Colors \succ Group Left Cosets. The result should be the same as your previous identification. What the program did was take the subgroup generated by all of the highlighted elements (i.e. the 5) and then it finds each coset and colors all the elements of the coset the same color. As with the grouping of elements each coset is grouped so changing the color of one element of the coset will automatically change the colors of each element of that coset.

So why did we have you do it the long way? It always helps to know what the program is doing.

3. Repeat the previous exercise with the subgroup $\{0, 2, 4, 6, 8\}$. You should determine what the cosets are

and what the quotient group is isomorphic to. Then repeat the color identification by cosets. Redraw the triangle and then answer the questions from the last exercise using your new image.

4. Using your answers for the previous exercises, explain how the subgroup lattice of \mathbb{Z}_{10} relates to the appearance of Pascal's triangle mod 10.
5. Repeat the above analysis for all of the non-trivial subgroups of \mathbb{Z}_{12} .