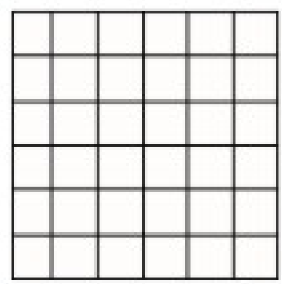
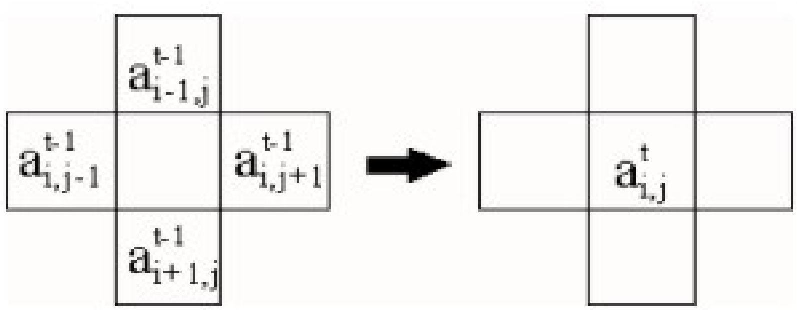
PascGalois Project

Two Dimensional Automata

In this project we will consider a finite two dimensional dynamical system generated using group multiplication from a finite group $(G, *)$. We begin with a finite $n \times n$ checkerboard grid:

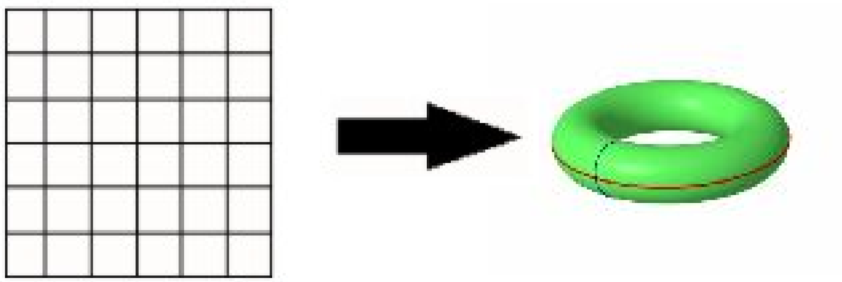


Each of the n^2 squares can be thought of as "cells" that, at any given time t, can hold some value from G. Let $a^t_{i,j}$ denote the contents of the cellin the i^{th} row and j^{th} column at time t. So $a^t_{i,j} \in G$. At time $t = 0$ the arrangement of cell values is considered the initial state (or initial condition) of the system. The grid is then updated in discrete time steps according to the local rule $a^t_{i,j} = a^{t-1}_{i,j-1} * a^{t-1}_{i-1,j} * a^{t-1}_{i,j+1} * a^{t-1}_{i+1,j}$:



The resulting dynamical system is an example of what is called a 2-D cellular automata. Note that this definition makes sense for the interior of the grid. However, there is a problem with the cells adjacent to the boundaries since those cells are not surrounded on all sides by neighboring cells. To fix this, we identify the top and bottom boundaries. Likewise we identify the two side boundaries. This causes all four corners to be identified to a single point.

The square grid is now wrapped into the surface of a torus so that every cell is surrounded by the same number of neighboring cells:



PascalGT will let you choose an initial state and then animate the time evolution of the system. In this project you will investigate how the choice of group, initial state, and grid size affects this evolution. Some of the exercises below are intentionally open-ended. Hopefully you will explore many examples and make some new conjectures of your own!

Exercises:

1. Using PascalGT select Z_4 as the group and then construct a 2-D automata using a 64×64 grid with initial state a one point square. This will place a 1 in one cell of the grid and zeros in all other cells. Animate the system until you understand the long-term behavior of the automata. Do all four group elements appear as cell values during the process? Does the automata ever enter a sequence of time frames where only a subset (or subgroup) of Z_4 appears as cell values?

2. RepeatExercise 1 using a 65×65 grid. What are the similarities between the two evolutions? What are the differences? Try other square grid sizes of side length 32, 34, 128, and 130. Can you make any conjectures based on these animations? Do they tell you anything about sensitivity to initial conditions?

3. In the previous two exercises, cell values were taken from abelian groups. Now let us try a nonabelian example. Repeat Exercises 1 and 2 using the dihedral group D_4. As before, note any similarities and differences that you find.

4. Experiment with different groups and grid sizes of your choice. You could try (but not necessarily) $Z_2 \times Z_2$, $Z_2 \times Z_3$, the quaternion group, D_5, Z_6, etc.

Make sure to use groups that have a variety of different structural properties. See what interesting conjectures you can find of your own. Can you prove (find a counter example) for any of them? Note: This is a great way to start an undergraduate research project!