PascGalois Project 1

Pattern Recognition and Pascal's Triangle Modulo n

You may be familiar with Pascal's triangle and its construction using the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$:

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & \vdots & & \vdots & &
\end{array}
$$

Using the fact that $\binom{n}{0} = \binom{n}{n} = 1$ for all $n \geq 0$, we see that Pascal's triangle has 1's down both sides of the triangle. Upon computing the rest of the entries we obtain the following:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & \vdots & & & & \vdots & &
\end{array}
$$

It is clear that any entry in the interior of the triangle is the sum of the two entries above it. This follows from the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. You may also have noticed that each row is symmetric, that is, a palindrome. This follows from the fact that $\left(\begin{array}{c} n \\ k \end{array}\right) = \left(\begin{array}{c} n \\ n-k \end{array}\right)$. The binomial theorem also states that the n^{th} row of Pascal's triangle gives the coefficients of the expansion for $(a+b)^n$:

$$(a+b)^n = \left(\begin{array}{c} n \\ 0 \end{array}\right) a^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) a^{n-1}b + \left(\begin{array}{c} n \\ 1 \end{array}\right) a^{n-2}b^2 + \cdots + \left(\begin{array}{c} n \\ n-1 \end{array}\right) ab^{n-1} + \left(\begin{array}{c} n \\ n \end{array}\right) b^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) a^k b^{n-k}.$$

In this exercise we will study a modified version of this triangle called Pascal's triangle modulo n. Pick an integer $n \geq 2$. Then reduce each entry of the triangle modulo n. Note that any entry in the interior of this new triangle is the sum mod n of the two entries directly above it. Also note that some very interesting patterns emerge inthis revised triangle. Of course these patterns depend upon which positive integer you picked initially. Since this construction can be time consuming to do by hand, use the program PascalGT to do it for you. PascalGT assigns to each of the values $0, 1, \ldots, n-1$ a unique color. See the PascalGT section in the Appendix for instructions on how to draw these triangles.

Exercises:

1. Construct Pascal's triangle modulo 2. What general pattern do you observe?

2. Repeat this for triangles modulo n, $n = 3, 4$, and 5. Compare all four triangles constructed thus far. Does any one of them have a quality or qualities that seem different from the other 3?

3. Experiment with Pascal's triangle mod n for $n = 6, 7, 8, 9$, and 10. How does n being prime or composite affect the appearance of the corresponding triangle?

Now we are going to interpret these triangles in terms of abstract algebra using the machinery of group theory. Recall that addition mod n is the group multiplication for the cyclic group Z_n. Think about the subgroups of Z_n, in particular for the values of n listed above. How many subgroups does Z_n have? List them for $n = 2, 3, \ldots, 10$.
You may also be familiar with the *order* of a group element. If G is a finite group and $a \in G$, then the order of a is the least positive integer n such that

$a^n = e$, where e is the identity of G. What can you say about the orders of elements from Z_n? List all the possible orders for $n = 2, 3, \ldots, 10$.

4. Is there any correlation between the lattice of subgroups of Z_n and the complexity of the corresponding mod n triangle? How does this relate to the factorization of n into prime factors? (*Hint*: Consider the number of relatively prime factors for each factorization).

5. By now you have studied the concept of *closure* and how each subgroup of a given group is a closed subset of that group under the group operation. Look for regions in the mod n triangles that only contain elements from a given subgroup. What is the shape of these regions? How does the concept of closure explain this shape?

6. Consider the triangles for Z_p where $p < 10$ is a prime. What can you say about the orders of the nonzero elements in these groups? Notice that the corresponding triangles have an outward growing self-similarity analogous to the inward self-similarity found in many fractals. How many rows must you "zoom out" to see this self-similarity in each triangle? Can you make a conjecture relating this row scaling factor to the orders of the non-zero group elements?

<u>Note</u>: If you are unfamiliar with fractals and the concept of self-similarity, your instructor may provide a brief overview or appropriate references.