PascGalois Project 2

Pattern Recognition Using Dihedral Groups

We are now ready to generalize the construction for Pascal's triangle mod n. Motivated by the fact that addition mod n is the group multiplication for the cyclic group Z_n, we now take any finite group G and let $a, b \in G$. A *PascGalois triangle* is formed by placing a down the left side of an equilateral triangle and b down the right. An entry in the interior of the triangle is determined by multiplying the two entries above it using the group multiplication. Of course, if G is nonabelian then one must specify a left or right multiplication. We denote this PascGalois triangle by (P_G, a, b). When a and b are clear from context we denote it simply by P_G. Note that Pascal's triangle mod n is $(P_{Z_n}, 1, 1)$. We will denote this simply P_{Z_n}. The construction of (P_G, a, b) is the following:

$$
\begin{array}{ccccccccc}
 & & & & a & & & & \\
 & & & a & & b & & & \\
 & & a & & ab & & b & & \\
 & a & & a^2b & & ab^2 & & b & \\
a & & a^3b & & a^2bab^2 & & ab^3 & & b \\
 & & \vdots & & & & \vdots & &
\end{array}
$$

Note that the a on top of the triangle is arbitrary and only appears for aesthetics. As a mathematical structure, P_G really begins at the second row. The word PascGalois comes from "splicing" Pascal with Galois. Galois (1811-1832) pioneered the study of groups while working with the theory of equations. Like Pascal's triangle mod n, PascGalois triangles can have many interesting patterns and self-similar properties. One of the goal's of this project, and the subsequent ones as well, is to understand these patterns in terms of group structure. Hopefully, studying the plethora of patterns within these triangles will provide you with a mechanism to visualize many of the fundamental concepts from abstract algebra.

In this project we will consider the dihedral groups D_n, $n \geq 3$ and introduce the concept of a $p-$group. Dihedral groups are the symmetry groups of regular polygons. D_3 is the symmetry group of an equilateral triangle, D_4 is the symmetry group of a square, D_5 is the symmetry group of a regular pentagon, and so on. If M is an n sided regular polygon, then M has n rotational symmetries and n reflectional symmetries. Hence the corresponding symmetry group has $2n$ elements. For example, let us consider the symmetry group of a square with corners labeled 1,2,3 and 4:

$$\begin{array}{cc} 1 & 2 \\ & \\ 3 & 4 \end{array}$$

By turning the square counterclockwise, we see that there are 4 rotational symmetries: $0°$, $90°$, $180°$, and $270°$. Of course $360°$, $450°$, ... are also symmetries, but they are equivalent to one of the four already listed. Note that we could also have done our rotations clockwise. We will use the convention that all the rotations will be counterclockwise since each clockwise rotation is equivalent to a counterclockwise rotation. Can you see why?

Wealso see that there are four axes for reflectional symmetries. There is a vertical axis that bisects the square. Reflecting about this axis interchanges corners 1 and 2 and also 3 and 4. There is an analogous horizontal axis. Reflection about it interchanges 1 and 3 and also 2 and 4. Finally, there are two axes passing through the diagonals of the square. One passed through corners 1 and 4. Reflection about this axis leaves corners 1 and 4 fixed but interchanges 2 and 3. You should be able to describe the last axis at this point.

We say a group G is a **p−group** if $|G| = p^n$ for some prime p. An equivalent definition is that G is a $p-$group if, given any $a \in G$, $o(a) = p^m$ for some nonnegative integer m. Here p is a fixed prime. For example the cyclic groups Z_{17}, Z_4, Z_{27}, and Z_{125} are all p-groups. However, Z_6 is not a $p-$group. Why? Before continuing to the problems below answer the following:

Question: D_n is a $p-$group if and only if ________________________________.

Exercises:

For the first few problems we will consider the dihedral groups D_3 and D_4.

1. Using PascalGT, draw the first 16 and 32 rows of (P_{D_3}, μ, σ) where σ is the 120° rotation and μ is a reflection. Note that this is the default setting in PascalGT (this corresponds to generators 1 and 3) so you do not need to change the generators once you have chosen D_3 as your group. Do you see any patterns? How do these images compare with the Pascal's triangle mod n pictures?

2. Now draw the first 64, 128, 256, and 512 rows of the same triangle. As you add more rows, what happens to the corresponding image? Are certain group elements "clumping together" in the triangles. If so, how does this relate to group structure? closure? If you have trouble seeing any patterns, try changing the color scheme (see PascalGT instructions - Color Schemes 5 and 14 may be good choices). Note: Later when we study quotient groups, it will be much easier to understand this triangle.

3. Repeat Exercises 1 and 2 for the D_4 triangle. Compare and contrast the images you see with the D_3 images. Can you make a conjecture regarding the differences between these two images?

4. Draw the first 64 and 128 rows of the D_5, D_6, D_7, and D_8 triangles. Compare and contrast the qualitative properties of these images. Does your conjecture in the previous exercise still seem to be true?

5. Earlier you should have given a necessary and sufficient condition for D_n to be a $p-$group. Which of the dihedral groups D_n, $3 \leq n \leq 8$, are $p-$groups? Does this seem to affect the appearance of the corresponding triangles?

6. Recall that two integers r and s are relatively prime if $\gcd(r, s) = 1$. Which of the dihedral groups in Exercise 5 have at least one pair of non-identity elements whose orders are relatively prime and which do not? Look at the subgroup triangles for these six PascGalois triangles. Is there any correlation between the existence of group elements of relatively prime order and the appearance of the subgroup triangles?