PascGalois Project 3

Building A Group With PascGalois Triangles

In this project you will build a group out of the PascGalois triangles. So far these triangles have been complicated structures with a plethora of internal detail. However, for the first three exercises you will think of each triangle as a mere "point" in a more abstract structure. The addition of these triangles will then be visualized using the superimpose option of PascalGT. This project should help you see that mathematical structures can often have many "layers". That is, a structure can have intricate internal detail when viewed from one perspective, but then appear point-like or atomic when viewed from another. Although PascGalois triangles are complex when viewed internally, they will appear to be point-like in the group you are about to form from them. Understanding the different layers of abstraction for various structures is one of the key goals of mathematics. It can also be a very powerful tool. This way of thinking is essential for every mathematics major.

Let G be any group and consider $\mathcal{G}=\{P_{a,b} \mid a, b \in G\}$. $\mathcal{G}$ can be thought of as the set of all possible PascGalois triangles that can be generated by the group G. In this lesson we will see that $\mathcal{G}$ can be given an algebraic structure of its own. If $P_{a,b}, P_{c,d} \in \mathcal{G}$ we can "multiply" $P_{a,b}$ and $P_{c,d}$ by defining $P_{a,b} * P_{c,d} \triangleq P_{ac,bd}$ where ac and bd are computed using the group operation of G.

Exercises:

1. Prove that under the operation * defined above, $\mathcal{G}$ forms a group. If G is abelian, show that $\mathcal{G}$ is also. You need to check that:
i) * is a binary operation on $\mathcal{G}$
ii) * is associative
iii) there is an identity element for $\mathcal{G}$ under *
iv) given any $P_{a,b} \in \mathcal{G}$, there exists an inverse $(P_{a,b})^{-1} \in \mathcal{G}$
v) $P_{a,b} * P_{c,d} = P_{c,d} * P_{a,b}$ for all $a, b, c, d \in G$, when G is abelian.

2. If G is finite, what can you say about the order of $\mathcal{G}$?

3. Prove that $\mathcal{G} \cong G \times G$.

4. How does the internal structure of each triangle (for example, self-similarity, level of organization, etc.) affect your solutions for Exercises 1, 2 , and 3?

5. We have defined our operation on $\mathcal{G}$ in terms of the generators down the sides of the triangle. Now we will investigate what would happen if we defined a superposition operation $\otimes$ where $P_{a,b} \otimes P_{c,d}$ is defined to be the triangle obtained by multiplying corresponding entries of $P_{a,b}$ and $P_{c,d}$ using the group operation of G. That is, if $g \in G$ is in the i^{th} row and j^{th} column of $P_{a,b}$ and $h \in G$ is in the i^{th} row and j^{th} column of $P_{c,d}$, then $gh \in G$ is in the i^{th} row and j^{th} column of $P_{a,b} \otimes P_{c,d}$. We need to be careful with this definition. First of all, is $\otimes$ well-defined? In other words, is $P_{a,b} \otimes P_{c,d}$ necessarily an element of $\mathcal{G}$? If not, then we do not have a binary operation. Even if $\otimes$ is a binary operation, is it the same as $*$? The purpose of the next three exercises is to answer these questions.

Draw $P_{1,2}$ and $P_{2,4}$ using Z_5 group multiplication. Use the Superimpose option in PascalGT to add mod 5 corresponding entries of the two triangles and then draw the result. Now, under our definition of addition for these triangles, $P_{1,2} * P_{2,4} = P_{3,1}$. Next have the program draw $P_{3,1}$. You should see that the $P_{3,1}$ triangle is the same as the triangle you obtained by superimposing the $P_{1,2}$ and $P_{2,4}$ triangles. Try this for other combinations as well. That is, superimpose other PascGalois triangles constructed over Z_5.

6. Prove that what occurred in the previous exercise happens in general for any abelian group G and $a, b, c, d \in G$. That is, show that by taking the n^{th} row of $P_{a,b}$ to the n^{th} row of $P_{c,d}$ produces the n^{th} row of $P_{ac,bd}$. What can you say about * and $\otimes$ in this case?

Hint: For any abelian group G and $a, b \in G$, the n^{th} row of (P_G, a, b) is

$$a \quad a^{\binom{n}{1}} b^{\binom{n}{0}} \quad a^{\binom{n}{2}} b^{\binom{n}{1}} \quad \cdots \quad a^{\binom{n}{n-1}} b^{\binom{n}{n-2}} \quad a^{\binom{n}{n}} b^{\binom{n}{n-1}} \quad b \,.$$

7. Now draw the D_4 triangles with generators 1,4 and then 2,5. Note that 1 and 2 (as PascalGT labels them) are rotations of 90 and 180 degrees, respectively. Use the Superimpose option in PascalGT to superimpose these two triangles. Is the superimposed triangle the same as $(P_{D_4}, 1 \circ 2, 4 \circ 5)$?

If the two triangles are the same, explain why. If they are different, explain why superposition for D_4 triangles does not work like it does for triangles generated by Z_n. What can you say about $\otimes$ in this case?