PascGalois Project 4

Direct Products and Automorphisms

Consider Pascal's triangle mod n. We have already observed that it is symmetric. This follows from the fact that Pascal's triangle is symmetric.

Question: What property of binomial coefficients causes this symmetry?

Another way we can interpret this symmetry is that reflecting the triangle about its central vertical axis induces the identity map $\iota : Z_n \to Z_n$ given by $\iota(m) = m$ for every $m \in Z_n$. That is, this reflection takes every color in the triangle to itself. Note that the identity map gives a (trivial) automorphism of Z_n. More generally, let G be *any* abelian group and $a, b \in G$. If we place a down the left side of the triangle and b down the right, then the group multiplication induces the following:

$$
\begin{array}{c}
a \\
a \quad\quad b \\
a \quad\quad ab \quad\quad b \\
a \quad\quad a^2b \quad\quad ab^2 \quad\quad b \\
\vdots \\
a \quad a^{\binom{n}{1}} b^{\binom{n}{0}} \quad a^{\binom{n}{2}} b^{\binom{n}{1}} \quad \cdots \quad a^{\binom{n}{n-1}} b^{\binom{n}{n-2}} \quad a^{\binom{n}{n}} b^{\binom{n}{n-1}} \quad b \\
\vdots
\end{array}
$$

Let V denote the central vertical axis that bisects P_G. If we reflect P_G about V then $a \longleftrightarrow b$ and

$$a^{\binom{n}{1}} b^{\binom{n}{0}} \longleftrightarrow a^{\binom{n}{n}} b^{\binom{n}{n-1}}$$

$$a^{\binom{n}{2}} b^{\binom{n}{1}} \longleftrightarrow a^{\binom{n}{n-1}} b^{\binom{n}{n-2}}$$

$$\vdots$$

$$a^{\binom{n}{j}} b^{\binom{n}{j-1}} \longleftrightarrow a^{\binom{n}{n-j+1}} b^{\binom{n}{j}}$$

Using the fact that $\binom{n}{k} = \binom{n}{n-k}$, we have $a^{\binom{n}{n-j+1}} = a^{\binom{n}{j-1}}$.

So in general $a^{\binom{n}{j}} b^{\binom{n}{j-1}} \longleftrightarrow a^{\binom{n}{j-1}} b^{\binom{n}{j}}$. In other words, reflection about V just interchanges the exponents of a and b.

In this lesson we will investigate what happens when $(P_{Z_n \times Z_m}, a, b)$, $a = (0,1)$ and $b = (1,0)$, is reflected about its central axis. Note that $Z_n \times Z_m$ is abelian so we can use the results from the previous paragraph.

Exercises:

1. Consider $(P_{Z_2 \times Z_2}, a, b)$, $a = (0,1)$ and $b = (1,0)$. Does reflecting this triangle induce a function $\phi : Z_2 \times Z_2 \to Z_2 \times Z_2$? That is, can one color in two different locations of the triangle ever be sent to two different colors? (Recall that a function must have a unique output for a given input). If a function is not induced can you say why? If a function is induced, is there anything special about this map (think in terms of preserving algebraic structure)?

2. Consider $(P_{Z_2 \times Z_3}, a, b)$, $a = (0,1)$ and $b = (1,0)$. Does reflecting this triangle induce a function $\phi : Z_2 \times Z_3 \to Z_2 \times Z_3$? If a function is not induced can you say why? If a function is induced, is there anything special about this map?

3.Now let us generalize the previous Exercises to $G = Z_n \times Z_m$, where n, m ≥ 2. Reflect $(P_G, (0,1), (1,0))$ about its central axis for various values of m and n. Try several combinations of n and m until you understand how these values affect the triangle. Give a conjecture as to when this reflection induces a function $\phi : Z_n \times Z_m \longrightarrow Z_n \times Z_m$? Note your answer will depend on the choices for n and m. When a set map is obtained, what special properties do you think it has?

4. (Optional) Give a proof for your conjecture(s) in the previous exercise. You may want to use the result discussed above that $a^{\binom{n}{j}} b^{\binom{n}{j-1}} \longleftrightarrow a^{\binom{n}{j-1}} b^{\binom{n}{j}}$ for any a, b from an abelian group G.