PascGalois Project 5

Quotient Groups 1

One of the most important results often covered in a first semester undergraduate algebra course is the Fundamental Homomorphism Theorem which establishes the relationship between homomorphic images and quotient groups. To understand this result, you must be comfortable with both homomorphic images and quotient groups. However, quotient groups can often be a difficult concept to understand. Building groups out of cosets (which are equivalence classes defined using a normal subgroup) can be unsettling when first studied. The purpose of this project is to visualize the coset identifications created when constructing a quotient group. Hopefully seeing graphical representations of these identifications will make it easier for you to see these equivalence classes as points in a quotient group. In Project 3 you treated a certain set of PascGalois triangles as points in a group. The internal detail of the triangles was ignored for this level of abstraction. The same idea applies to quotient groups. Even though each coset is has internal structure (recall that cosets are subsets that partitition the group), it is treated as a point in the quotient group.

Exercises:

1. Construct Pascal's triangle mod 10 with 64 rows, 125 rows, and 128 rows. Study the patterns you see very carefully. Notice that there seem to be "competing themes" within the triangle. That is, there appear to be two distinct patterns, one superimposed on top of the other. Based on the colors you see and the group elements to which they correspond, can you predict what may be causing this visual effect?

Note that {[0],[5]} is a subgroup of Z_{10}. Since every subgroup of an abelian group is normal, we know that coset addition is well-defined for $Z_{10}/\{[0],[5]\}$ and, hence, it is a group. Certainly $Z_{10}/\{[0],[5]\} \cong Z_5$. The cosets are {[0], [5]}, {[1], [6]}, {[2], [7]}, {[3], [8]}, and {[4], [9]}. Which coset acts as the identity for the quotient group? Which coset(s) generate the quotient group?

2. Go back to your computer drawing of Pascal's triangle mod 10. Using the color subsets options in PascalGT, re-color the elements of Z_{10} by identifying elements in a common coset with the same color. You will need 5 distinct

colors – the first color for $\{[0], [5]\}$, the second for $\{[1], [6]\}$, and soon. Re-draw the triangle with these color identifications. What do you see? Can you explain the observed pattern based on group structure and quotient groups?

3. Repeat the previous exercise with the subgroup $\{[0], [2], [4], [6], [8]\}$. You should determine what the cosets are and what the quotient group is isomorphic to. Then repeat the color identification by cosets. Re-draw the triangle and then answer the questions from the last exercise using your new image.

4.Using your answers for the previous exercises, explain how the subgroup lattice of Z_{10} relates to the appearance of Pascal's triangle mod 10.