PascGalois Project 6

Quotient Groups 2

This lesson is a continuation of the work you did with quotient groups in Project 5. There you considered chains of subgroups of the form $G \geq H \geq \{e\}$, where e is the identity of G. Note that chains of this form have length 2. You may wonder what happens if one considers longer chains of subgroups of the form $G \geq H_1 \geq H_2 \geq \cdots \geq H_n \geq \{e\}$. We will consider such chains in this lesson. Note that if H_{i+1} is a normal subgroup of H_i for $i = 1, ..., n-1$ and H_1 is a normal subgroup of G, then we may construct quotient groups G/H_1, $H_1/H_2, \ldots, H_{n-1}/H_n$, and $H_n/\{e\} \cong H_n$. In this case the chain is called a *subnormal series* and written $G \rhd H_1 \rhd H_2 \rhd \cdots \rhd H_n \rhd \{e\}$ (in general, $H \lhd G$ means H is a normal subgroup of G).

For our first example, we will consider the subnormal series $Z_8 \rhd \{0, 2, 4, 6\} \rhd \{0, 4\} \rhd \{0\}$ (since Z_8 is abelian, we know that each subgroup in the chain is a normal subgroup of the proceeding group).

Exercises:

1. Using PascalGT draw the first 64 rows of Pascal's triangle mod 8. Look at the image you just created. Find the {0,2,4,6} subgroup triangles in your image. Next consider the quotient group $Z_8/\{0, 2, 4, 6\}$. Determine the cosets of this quotient. Using the color options in PascalGT, re-color the elements of Z_8 by identifying elements in a common coset with the same color. You will need a distinct color for each coset. Redraw the triangle and describe the picture that you obtain.

2. Go back to the $\{0, 2, 4, 6\}$ subgroup triangles that you identified in Exercise 1. Identify the $\{0, 4\}$ subgroup triangles *within* the $\{0, 2, 4, 6\}$ subgroup triangles. Determine the cosets for $\{0, 2, 4, 6\}/\{0, 4\}$. Using the *Color Subsets* options in PascalGT, re-color the elements of a $\{0, 2, 4, 6\}$ subgroup triangle by identifying elements in a common coset with the same color. Re-draw the subgroup triangle and describe what you see.

The next example we will consider is the dihedral group D_4, the symmetry group of a square. You may want to review Project 2 regarding dihedral groups. Let us denote the corners of the square as follows:

1 2

3 4

Then the elements of D_4 in cycle notation are as follows:

$r_0 = (1) \qquad r_1 = (1342) \qquad r_2 = (14)(23) \qquad r_3 = (1243)$

$\mu_1 = (12)(34) \qquad \mu_2 = (13)(24) \qquad \mu_3 = (23) \qquad \mu_4 = (14)$

Certainly r_0, r_1, r_2, and r_3 are counterclockwise rotations of $0°$, $90°$, $180°$, and $270°$, respectively. Then r_0 is the identity of D_4. The last four elements correspond to the reflectional symmetries of the square.

3. Using PascalGT draw the first 64 rows of $(P_{D_4}, 4, 1)$. Note that the program labels the rotations (from smallest to largest) as 0, 1, 2, and 3. Likewise the reflections are 4, 5, 6, and 7. Now look at the image you just created. Find the rotational subgroup triangles. Caution: It may appear that the subgroup triangles only contain 2 of the 4 rotations. However, look more closely. There should be some subgroup triangles containing all 4 rotations. Describe how each of the 4 rotations is distributed within the subgroup triangle.

4. Note that $H_1 = \{r_0, r_1, r_2, r_3\}$ is a normal subgroup of D_4 (why?). So D_4/H_1 is a quotient group. What is the order of this group? What are the cosets? Which coset acts as the identity? Using the color options in PascalGT, re-color the elements of D_4 by identifying elements in a common coset with the same color. You will need a distinct color for each coset. Redraw the triangle and describe the picture that you obtain. Have you seen it before?

5. Explain why the chain $D_4 \triangleright H_1 \triangleright H_2 \triangleright \{r_0\}$, where $H_2 = \{r_0, r_2\}$, is a subnormal series. We have already considered the quotient D_4/H_1 in the previous exercise. Go back to the H_1-subgroup triangles that you identified in Exercise 1. Identify the H_2-subgroup triangles *within* the H_1-subgroup triangles. Determine the cosets for H_1/H_2. Using the color options in PascalGT, re-color the elements of H_1 by identifying elements in a common coset

with the same color. Re-draw the triangle and describe what you see. Does anything occur with this subnormal series and corresponding triangle that did not occur with the Z_8 case from Exercises 1 and 2?

6. Consider the subgroup $H = \{r_0, r_2\}$ of D_4. You should check that this is indeed a normal subgroup and that $D_4 \triangleright H \triangleright \{r_0\}$ is a subnormal series (note that $H = \{0, 2\}$ in the notation of the program). Determine the cosets of D_4/H. This quotient is isomorphic to a group you have seen before. Which group is it? Using the color options in PascalGT, re-color the elements of D_4 by identifying elements in a common coset with the same color. Re-draw the triangle and describe what you see. Is this a triangle you have seen before?

Now consider D_8, the symmetry group of a regular octagon.

7. Find a subnormal series $D_8 \triangleright H_1 \triangleright H_2 \triangleright H_3 \triangleright \{r_0\}$ where $|H_1| = 8$, $|H_2| = 4$,and $|H_3| = 2$. Perform the same type of analysis that you did for D_4 in Exercises 3,4 and 5 on this subnormal series. Do you obtain similar results?

8. Can you find a normal subgroup $H \triangleleft D_8$ such that $D_8/H \cong Z_2 \times Z_2$? If so, redraw the D_8 triangle by identifying colors according to cosetsso that your image looks like the Klien-4 triangle.