Connectivity of n-Tuple Graphs

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Abstract

- Let G be a graph and $n \leq |V(G)|$ a positive integer. The *n-tuple graph of* G, denoted by $U_n(G)$, has all n-element subsets of $V(G)$ as its vertices, and two vertices are adjacent if and only if they have exactly $n-1$ elements in common and the remaining ones are adjacent in G.
- We determine formulas for the connectivity of $U_n(C_m)$ and $U_n(K_{s,t})$, and define criteria for the connectivity of $U_n(T)$.

N-Tuple Graph Definition

The *n-tuple graph of* G, denoted $U_n(G)$, has

- All n-element subsets of $V(G)$ as vertices
- An edge between two vertices if and only if
  - The vertices have exactly $n-1$ elements in common.
  - The remaining two elements are adjacent in G.

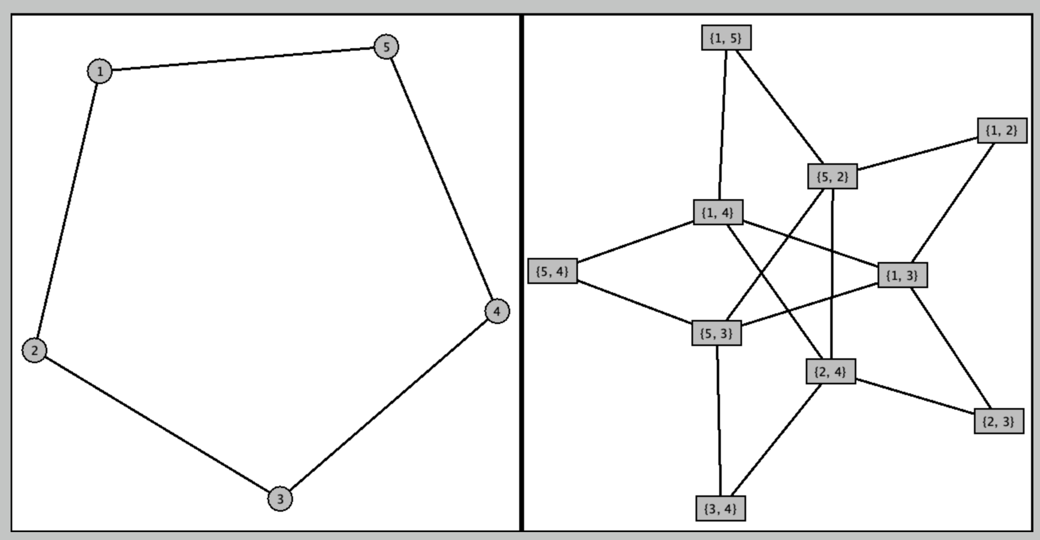

Figure: C_5 and $U_2(C_5)$

Prior Results

Theorem 1.5: Lower Bound of $\kappa(U_n(G))$
Given a simple finite graph G with minimum degree δ, the n-tuple graph $U_n(G)$ is $\max\{\delta, (n-1)(\delta-n+1)+1\}$-connected. [1]

- This bound is not the connectivity of $U_n(G)$; counterexample below, where $\max\{\delta, (n-1)(\delta-n+1)+1\} = 3$ but connectivity is 8.

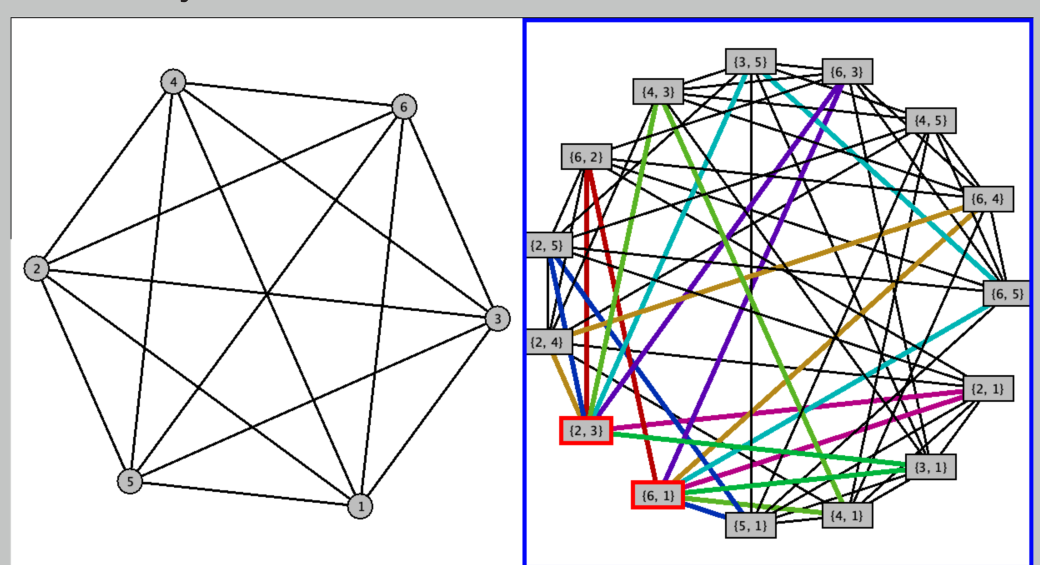

Figure: K_6 and $U_2(K_6)$

Proof Preliminaries

Lemma 2.2: Condition for $\kappa(U_n(G))$
The existence of k internally disjoint paths between arbitrary *pseudo-adjacent* $X, Y \in V(U_n(G))$ implies k-connectedness of $U_n(G)$. [1]

- Connectivity is determined by finding equality between an upper and lower bound.
- A common **upper** bound is minimum degree.
- A common **lower** bound is internally disjoint paths between pseudo-adjacent vertices, which is sufficient by Lemma 2.2.

Connectivity of $U_n(K_p)$

Theorem: $\kappa(U_n(K_p)) = n(p-n)$

- This was proven by Wright [2] using a counting argument.
- We proved it again by finding that the minimum degree enumerating paths between pseudo-adjacent vertices.

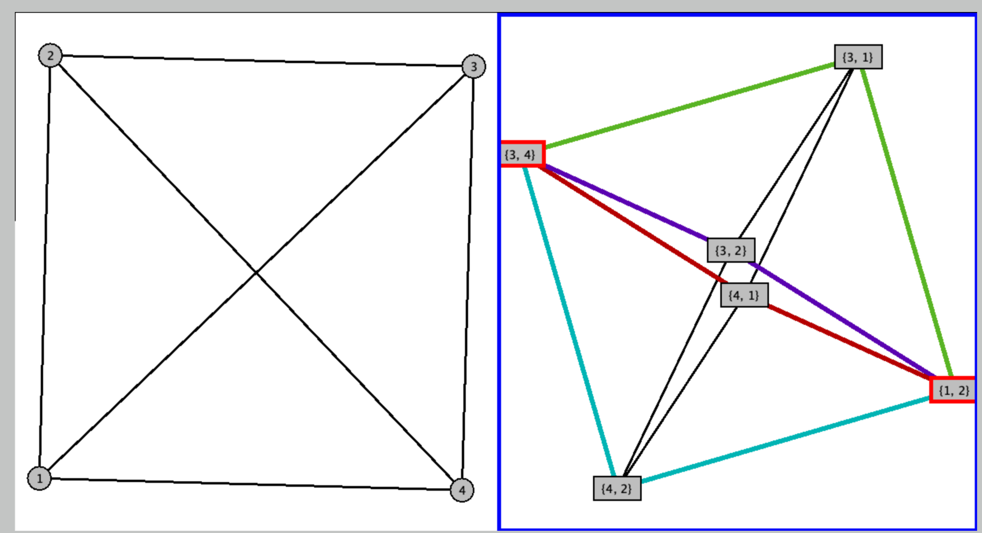

Figure: K_4 and $U_2(K_4)$

Connectivity of $U_n(C_m)$

Theorem: $\kappa(U_n(C_p)) = 2$

- $\kappa(U_2(C_5)) \leq \delta(U_2(C_5)) = 2$
- $\kappa(U_2(C_5)) \geq \min_{X,Y \in V(U_2(C_5))} p(X,Y) = 2$. The two paths are called "upward" and "downward" shifting paths.

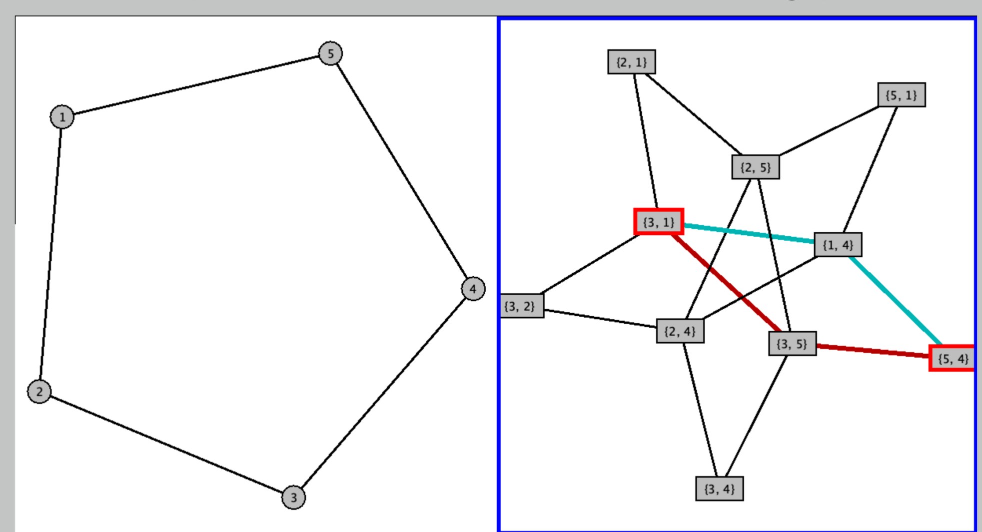

Figure: C_5 and $U_2(C_5)$

N-Tuple Graphs of Trees

Theorem: For a tree T of order p, and positive integer n such that $1 \leq n \leq p-1$, we have

$$\delta(U_n(T)) \leq n. \tag{1}$$

The Big Leaf Conjecture

Define a "big leaf" of a tree T to be a subtree $R \subseteq T$ such that $T \setminus R$ is not disconnected.

Conjecture (Big Leaf): Given a tree T, $\kappa(U_n(T)) = 1$ if and only if T has a big leaf of order n.

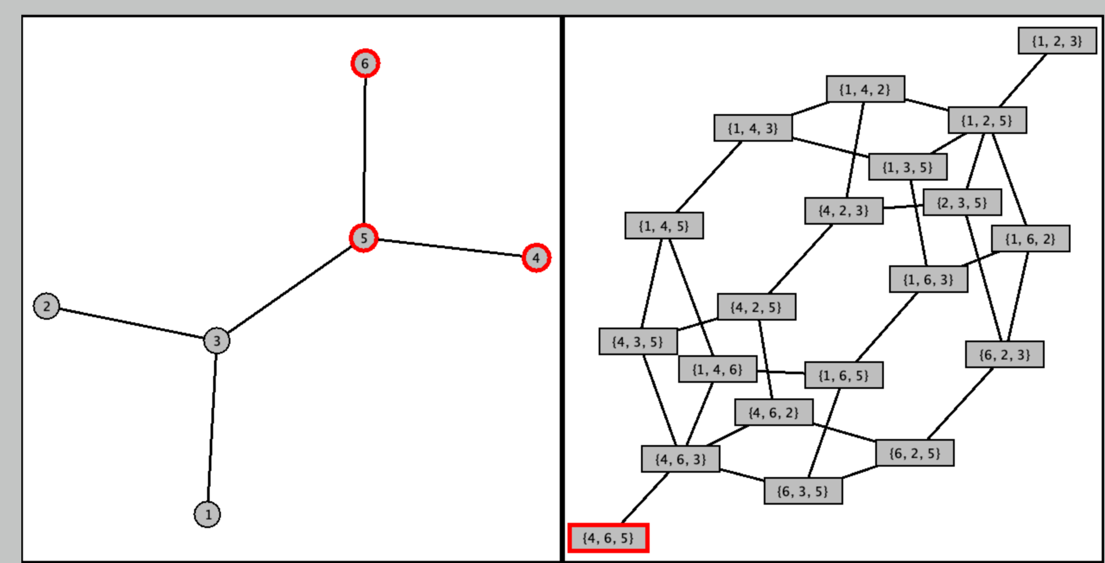

Figure: A big leaf of order 3

The backwards direction of the Big Leaf Conjecture has been shown.

- Assume T has a big leaf $R \subset T$ of order n. There will always be exactly one edge $rt \in E(T)$ such that $r \in R$ and $t \in T \setminus R$.
- R is a vertex in $V(U_n(T))$, and is always adjacent to $\{t\} \cup T \setminus \{r\}$.
- The existence of any other neighbor of R contradicts T being acyclic, so R has degree 1 in $U_n(T)$. Thus $\kappa(U_n(T)) = 1$.

It can also be shown that any vertex in $U_n(T)$ that has degree 1 is a big leaf in T.

Connectivity of $U_n(T)$

Conjecture: $U_n(T)$ is maximally connected.

We suspect this conjecture is true, and it would be more than enough to complete the forwards direction of the Big Leaf Conjecture.

Connectivity of $U_n(K_{s,t})$

Theorem: Let $N = \min\{n, s+t-n\}$

$$\kappa(U_n(K_{s,t})) = \begin{cases} \left\lceil \frac{(t+s)}{2} n - \frac{n^2}{2} - \frac{(t-s)^2}{8} \right\rceil, & N \geq \frac{|s-t|}{2} \\ N(\min\{s,t\}), & N < \frac{|s-t|}{2} \end{cases}$$

- This claim was proven analogously to the case of cycles, in which the desired value for $\kappa(U_n(K_{s,t}))$ is shown to be an upper bound as this value is the minimum degree of the graph.
- It is then shown to be a lower bound by proving that between all pseudo adjacent $X, Y \in V(U_n(K_{s,t}))$, there exist no fewer than that value of internally disjoint X-Y paths.
- It is worth noting that for $n, m \in \{1, \ldots, \lfloor \frac{s+t}{2} \rfloor\}$, $n > m$ implies $\kappa(U_n(K_{s,t})) > \kappa(U_m(K_{s,t}))$

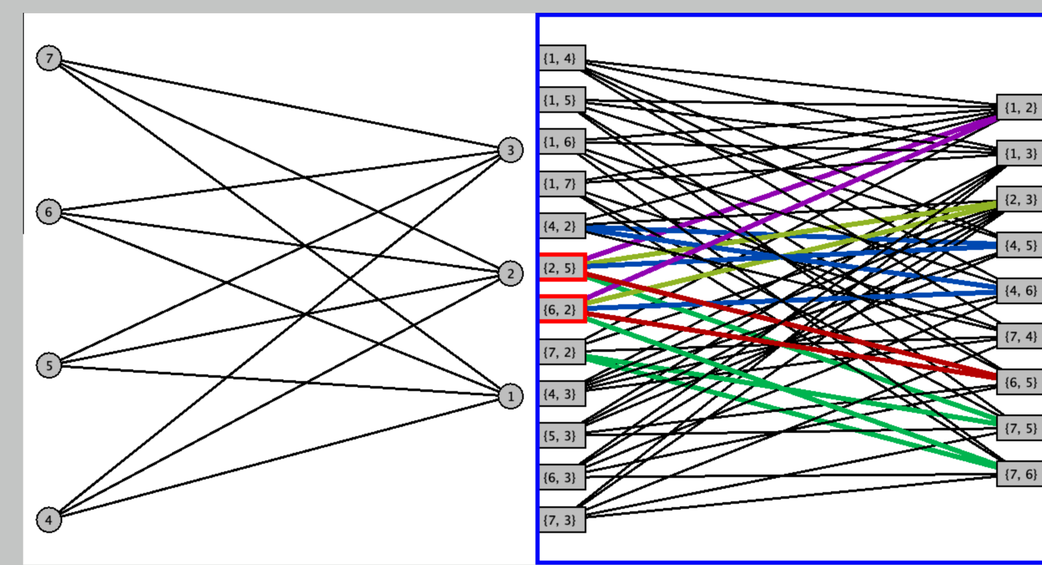

Figure: $K_{3,4}$ and $U_2(K_{3,4})$

Visualization Software

All graphics on this poster were created with *n-Tuple Grapper*, a visualization software for n-tuple graphs developed in Java for this research project. The program provides a user interface to dynamically create and view graphs and their n-tuple graphs. It can be found here:
`https://github.com/deplatt/n-tuple-grapper`

References

[1] R. Goodnow, A. Halperin, H. Wang, and B. Wu.
Proof of a conjecture on the connectivity of n-tuple graphs.
2024.

[2] V. Wright.
n-tuple vertex graphs.
Master's thesis, Emory University, 1992.

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