Introduction >>

The Connections Standard set forth in Principles and Standards for School Mathematics (NCTM, 2000) reminds us: "When students can connect mathematical ideas, their understanding is deeper and more lasting" (p. 64). Unfortunately, many precalculus students learn trigonometry without making connections to related concepts they learned during their year of geometry—tangent segment, secant segment, Pythagorean Theorem, similar triangles, and so forth. This article emerged from recent experiences in attempting to make such connections when teaching a high school precalculus course.

Applets 1 and 2 were created from Lesser (2004), figures 1 and 2, respectively, where tangent and secant lines are added to a unit circle. These applets allow students to make and explore interactive connections not only with the sine and cosine functions, but also with the four other basic trigonometry functions: tangent, cotangent, secant, and cosecant. Through the use of these applets, students of diverse learning styles may gain more intuition about the values of each function. In each applet, by dragging point A around the circle, they will "see" the values each function takes on as the angle changes and thus make connections to major inequalities and identities.
An alternative way in which we could have represented the six trigonometric functions with segment lengths is by modifying applet 1 by extending radial ray $OC$ and drawing a tangent line through $(0,1)$. Let $E$ denote the point $(0,1)$, and let $F$ denote the intersection with ray $OC$. We can now identify six segments whose lengths represent the values of the six basic trigonometry functions of $\angle AOB$, or $\theta : \sin \theta = AB$, $\cos \theta = OB$, $\tan \theta = CD$, $\cot \theta = EF$, $\sec \theta = OC$, $\csc \theta = OF$. This modified figure resembles exercise 5 in Usiskin et al. (2003, p. 449).

Students may explore adapting the applet representation to model a particular version of the Ferris Wheel Problem, which can be generalized as follows:

Model the height above ground of a point on a Ferris wheel, the radius of which is $r$ feet and the bottom of which is $a$ feet above ground, as a function of time $t$, given that a revolution takes $b$ seconds, the wheel turns counterclockwise, and people get on at the bottom of the wheel.

Sources of Ferris wheel problems with specific numbers include COMAP (2002, pp. 256–58), Campbell, Kemp, and Zia (2006), and www.nctm.org/high/asolutions.asp?ID=365. The Ferris Wheel applet, applet 3, allows students to see clearly the essential part of the previous applets. (Calculus teachers may want to read “Mathematical Lens” in the November 2005 issue of Mathematics Teacher.)

To move toward a more general solution to the Ferris Wheel problem, let’s consider the height of point $A$ in applet 1, where we have previously noted that the $y$-coordinate of point $A$ is $\sin \theta$. Now, giving the Ferris Wheel a radius of $r$ feet changes the $y$-coordinate to $r \sin \theta$. Then, moving the center of the Ferris Wheel from the origin to the point $(0, r + a)$ to give the bottom of the wheel the necessary clearance of $a$ feet above the ground changes the height function to $h(\theta) = r + a + r \sin \theta$. People commonly board a Ferris Wheel at its bottom point (we’ll call it $E$). Thus the important angle is really $\angle EOA$; we’ll call it phi ($\phi$), which is simply $\theta + (\pi/2)$. So the height function becomes $h(\phi) = r + a - r \cos \phi$. If a revolution of $2\pi$ happens in $t = b$
seconds, then \( \varphi = (2n/b)t \). By substitution, the height function becomes \( h(t) = a + r - r(\cos((2n/b)t)) \).

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In applet 2, we have an only slightly more involved diagram that contains triangle segments whose lengths represent all six basic trigonometry functions: sine, cosine, tangent, cotangent, secant, and cosecant. Recall that the very root of the word trigonometry invokes the measurements (angle and side) of a triangle: the Greek words trigon and metron mean “triangle” and “measure,” respectively. As Schwartzman (1994) adds: “Historically speaking, the triangular approach to trigonometry is ancient, whereas the circular approach now taught in our schools is relatively recent” (p. 228). Here, circle O is a unit circle (so radius OA has length 1) and line PQ is tangent to the circle at A. Therefore, for central angle AOB, we know the following:

\[
\sin \angle AOB = \frac{AB}{OA} = \frac{AB}{1} = AB \\
\cos \angle AOB = \frac{OB}{OA} = \frac{OB}{1} = OB \\
\tan \angle AOB = \tan \angle AOP = \frac{PA}{OA} = \frac{PA}{1} = PA \\
\sec \angle AOB = \sec \angle AOP = \frac{OP}{OA} = \frac{OP}{1} = OP \\
\]

To be able to identify the segments whose lengths correspond to the cotangent and cosecant, we will need to show that \(\angle PQO\) is congruent to \(\angle POA\). This will require another connection to geometry: similar triangles. Because \(\angle QOP\) and \(\angle AOP\) are right angles and, therefore, congruent and because \(\angle OPQ\) and \(\angle APO\) are identical angles and also congruent, we conclude, on the basis of the Angle Angle Theorem, that \(\triangle QOP \sim \triangle OAP\). And now the third pair of corresponding angles—\(\angle PQO\) and \(\angle POA\)—must also be congruent.

Using this fact, together with our observations that \(\angle AOB\) and \(\angle AOP\) are the same angle and that \(\angle OPQ\) and \(\angle OQA\) are the same angle, we can determine the last two trigonometry functions as follows:

\[
\cot \angle AOB = \cot \angle AOP = \cot \angle OPQ = \cot \angle OQA = \frac{QA}{OA} = \frac{QA}{1} = QA \\
\]

Similarly, \(\csc \angle AOB = \csc \angle AOP = \csc \angle OQA = \frac{OQ}{OA} = \frac{OQ}{1} = OQ\)

Applet 2 allows students to explore all three Pythagorean Identities (see fig. 1)
Because the length of the cosecant segment \( OQ \) is always at least the length of the unit radius \( OR \), applet 2 helps make it easy to visualize how the magnitude of \( \csc \theta \geq 1 \), with equality for odd integer multiples of 90 degrees. We can also see that when the angle passes an odd integer multiple of 45 degrees, there is a change in whether \( \tan \theta \) or \( \cot \theta \) is larger in magnitude. The same can be said for \( \cos \theta \) or \( \sin \theta \).
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**Applet 1 >>**

Let’s begin with applet 1, the simpler applet. In the unit circle $O$, we see that the secant of central angle $COD$ is the ratio of $OC$ to $OD$. As the radius of the unit circle, $OD = 1$, so this ratio is the length $OC$, which is a segment on a secant line to the circle. The tangent of $\angle COD$ is the ratio of $CD$ to $OD$. This ratio is $CD$ since $OD$ is a unit circle radius, which is a segment on the tangent line to the circle at point $D$. This definition of tangent connects to students’ prior knowledge from geometry class, in which they learned that a tangent line intersects a circle in exactly one place, while a secant line intersects a circle in two places.

Geometry and trigonometry courses typically involve figures that include triangle $AOB$ and makes connections to the sine and cosine functions. Few classes automatically include opportunities to explore these functions with figures that also include triangle $COD$.

Let’s describe some of the benefits of using applet 1 with students. Right triangles $OAB$ and $OCD$ demonstrate the Pythagorean identities $\sin^2 \theta + \cos^2 \theta = 1$ and $\tan^2 \theta + 1 = \sec^2 \theta$, respectively. Because the length of the secant segment is always at least the length of a unit radius, the applet helps make it visually more clear how the magnitude of $\sec \theta \geq 1$, with equality for a zero angle or a straight angle. Similarly, the applet helps make it easier to note how the magnitude of $\tan \theta$ is greater than or equal to the magnitude of $\sin \theta$, with equality for a zero angle. And we can also see how $\tan \theta$ is actually the slope of $OC$, because the “run” is a unit length.