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# Hitting the Bull's A Dart Game Simulation Using Graphing Calculator Technology

HAT DO PLOTTING POINTS, MAKING decimal representations, graphing circles, using a random number generator on a calculator, calculating the Pythagorean theorem, studying probability, and throwing darts have in common? We did not know the answer to that question either until we accidentally discovered an activity that incorporates all these elements.

One problem that students have with mathematics is that they often view the topic as a series of unrelated ideas. Sometimes they are aware that they have to know one concept to move to the next, but



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what is done in geometry is not necessarily related to anything in algebra. This failure to recognize mathematical connections limits students' comprehension and understanding of mathematics. One way to counter this problem is by using activities that show multiple mathematical relationships.

This activity started as a simple lesson to teach students how to use the random number generator on the calculator to generate ordered pairs, graph the ordered pairs, and see how many of their points landed in a bull's-eye circle. Our objective was to use technology to teach a probability lesson. Over the past several years, based on suggestions from students and teachers, the activity has grown to incorporate a variety of concepts and skills. The general premise is still the same: "How many 'darts' hit a circular dart board with a radius of 1 unit?" The activities surrounding the simulated game of darts have grown to be quite rich.

## **Throwing Darts**

MANY STUDENTS ARE FAMILIAR WITH THE GAME of darts. Since we cannot bring real darts into the classroom for students to use, we were anxious to



find out how "pointless" darts would work. That is why this activity initially sounded so novel.

As noted earlier, we begin with a calculator and a paper dart board. The initial tasks are to generate pairs of random numbers and then have students graph those ordered pairs by hand on their dart board.

In our initial attempt to do this activity, we used the dart board shown in **figure 1**. It contains no markings so students have to be able to estimate or accurately measure the tick marks for their graphs. For students to be able to graph ordered pairs on that dart board, the values had to be between -1 and 1 for both the *x*-and the *y*-coordinates. This parameter elicited the first problem that students had to solve: How do we use the random number generator on the calculator?

#### **Generating Pairs of Random Numbers**

WE STARTED BY EXPLAINING THE RANDOM NUMber generator on the graphing calculator, emphasizing that it is not truly random. Therefore, to ensure that each student gets different results, each calculator needed to be seeded with a different value. If you have a classroom set of calculators, you can do this ahead of time by entering a different number on each calculator, then hitting the store key and the random number command (see **fig. 2**). The random number command



Fig 1 A blank bull's-eye for student use

is found by pressing the MATH key, then going to the probability menu (PRB).

The goal is to have students generate ordered pairs with *x*- and *y*-coordinates between -1 and 1 inclusive. However, the random number generator produces numbers between 0 and 1. We have students share ideas about how to find random numbers between -1 and 1, inclusive, using the calculator.

If the calculator only generates numbers between 0 and 1, we can set up the inequality 0<rand<1. Ask the students if this inequality will give us the ordered pairs that we need, and they correctly respond that it does not. Properties of inequalities tell us that multiplying each piece of the inequality by a positive number will not change the signs. If we multiply by 2, we get 0<2rand<2, which still does not produce the ordered pairs we need. However, by subtracting 1 from each piece of the inequality, we get -1<2rand-1<1. If students input 2rand-1 into their calculator, they will get a number in the desired range.

However, we want ordered pairs, not just one number. Texas Instruments graphing calculators will allow you to put a number after the "rand" command to generate more than one number at a time. To generate two numbers between -1 and 1, input 2rand(2)–1.

Although this can seem like a lot of work just to generate ordered pairs for students to graph, we now approach it as a teaching tool to introduce or refresh properties of inequalities, depending on the level of the students (**fig. 3**). We have students set their calculator mode to two decimal places so that the screen display is not too cluttered. However, if you want your students to have more practice with the concept of rounding, then let the calculator display the full decimal value and have students do the rounding themselves.

Now that students know how to generate a set of ordered pairs, we proceed with "throwing darts." Each ordered pair is treated as a "dart." By continuing to press ENTER, students continue to generate ordered pairs on the calculator screen. They then graph each ordered pair by hand on the paper bull's-eye. We usually have the students each graph 25 ordered pairs. Every student then determines how many of his or her 25 points are inside and how many are outside the circle. In other words, how many hit the bull's-eye and how many did not? As the activity has evolved, we have started using grid paper to make it easier.

#### **Experimental Probability Calculations**

A REVIEW OF PROBABILITY NOW ENTERS INTO the lesson. By recalling that the probability of an event is the number of ways to be successful divided by the number of ways that the event can happen, students determine that the probability of hitting the bull's-eye is the number of points inside the circle (success-



Fig. 2 An example of reseeding the random number generator

```
0<rand<1
0<2rand<2
-1<2rand–1<1
```

Fig. 3 Inequalities used in working with the random number command

ful) divided by 25 (the total number of points). Each student then computes his or her own experimental probability for hitting the bull's-eye.

Each student announces the number of points that landed inside the circle. The sum of these points is divided by the total number of darts in the classroom (number of students  $\times$  25). It is uncanny how often the experimental probability for a class is very close to the theoretical probability, which is what we discuss next in the lesson.

#### **Theoretical Probability Calculations**

By again recalling the definition for probability, we ask students to think about what the answer should be. Determining the number of ways to be successful now involves using geometry. The students realize that they do not have an actual number to use. To be successful still means landing in the circle, but it now requires finding the area for the circle. Most students point out that  $A = \pi r^2$  and that we have been dealing with a circle of radius 1. Finding the total number of ways for the event to happen (the denominator of the probability fraction) requires that students work carefully.

Because the students have been accustomed to dealing with the circle of radius 1, they are often tempted to answer too quickly about finding the area for the square that encloses the bull's-eye circle. As with the area of the circle, students are quick with the area formula for a square but have to realize that each side of the square is 2 and not 1.

Theoretically, the probability of a dart landing in the bull's-eye is

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\pi r^2}{l \cdot w} = \frac{\pi}{4} \approx .79.$$

As mentioned earlier, the number of times that the class experimental probability is close to this value is amazing. We have yet to conduct this activity with a group when the class value was more than  $\pm .03$  from the theoretical probability.

When we first started using this activity, we stopped here. After all, we had used the random number generator, graphed some points, talked about probability and statistics, and recalled some area formulas. Then a student asked if the graphing could be done on the calculator. It was an idea that intrigued us, so we explored that option and developed the following calculator extension.

#### **Calculator Darts**

WE KNEW THAT TO GRAPH ORDERED PAIRS OF numbers we would need to put values into lists and graph a scatter plot. We would also need to graph a circle to show where the points landed in relation to that circle. We tackled the ordered pairs first.

Graphing the ordered pairs by using lists requires a slightly different approach than generating ordered pairs to be graphed by hand. We need to get 25 values in List 1 and 25 values in List 2 and treat those as the *x*- and *y*-coordinates of 25 ordered pairs. The same principle for generating numbers between -1 and 1 still applies. However, to put 25 values into the two lists, use the following commands:

$$2$$
rand $(25)-1 \rightarrow L1$   
 $2$ rand $(25)-1 \rightarrow L2$ 

With the random numbers in the lists, a scatter plot can be graphed. The window needs to be set from -1 to 1 in each direction to simulate the square that encloses the bull's-eye. However, 25 points on a scatter plot mean nothing without a circle. Again, this presents a great opportunity to introduce or review the equation of a circle and how this has to be entered into the calculator (**fig. 4**).

However, when students press GRAPH and expect to see a recreation of what they did by hand, they are disappointed to see a figure that looks like an ellipse rather than a circle. Because the calculator screen does not have a 1:1 ratio for the *x*- and *y*-axes, the graph of the circle does not appear as students expect. The ratio of the *x*- and *y*-axes are actually in a 1:1.5 ratio, and the window settings must be adjusted accordingly so that the calculator will produce a graph in the shape of a circle. By performing the ZOOM Square command, students will see a circle with 25 points. The same process for finding experimental probability can be done at this point. Since the theoretical probability has already been computed, it is a quick way to compare it with the class experimental probability.

Plot1 Plot2 Plot3  

$$Y_1 \blacksquare J(1-X^2)$$
  
 $Y_2 \blacksquare -J(1-X^2)$   
 $Y_3 = \blacksquare$   
 $Y_4 =$   
 $Y_5 =$   
 $Y_6 =$   
 $Y_7 =$ 

Fig. 4 Equations for graphing a circle with radius 1 unit

We were very pleased with the outcome and felt a great sense of accomplishment for adding an extra bit of technology to the activity. However, a student asked a question that made us put more thought into what we were doing. A common question that had been asked when graphing by hand was this: "How can you tell whether your point should be inside the circle or outside the circle?" We had been telling students to simply guess whether the point was inside or outside the circle but decided that we really needed a better answer.

We began expanding the activity once again. Since we had the numbers in List 1 and List 2 to use as the ordered pairs, we could again take advantage of the calculator technology. This further use of technology also allowed the opportunity to talk about another mathematical topic, the Pythagorean theorem. How deeply you delve into this portion of the activity depends on the level of your students. You may wish to explore how and why using the theorem will work, have students explore a historical aspect by researching Pythagoras, or state the theorem and that it will be used to make decisions in the problem.

Understanding how and why the theorem works does not require a knowledge of trigonometry. We have used this extension successfully with some middle-grades students by explaining each step completely. Success with other students requires taking  $a^2 + b^2 = c^2$  on faith without explaining why it works.

To begin this segment of the activity, we pick a point on the circle and ask how far away it is from the center. Since we are working with a circle of radius 1, most students are able to answer quickly that a point on the circle is 1 unit away from the center.

With more advanced students and in-service teachers, we create a triangle using the two endpoints of the diameter and a third point on the circle and talk extensively about the Pythagorean theorem. With less advanced students, we point out that the Pythagorean theorem allows us to use the  $a^2 + b^2 = c^2$  relationship to help us determine whether points are inside or outside the circle. Since the circle has radius 1, then  $a^2 + b^2 < 1$  will be inside the circle and



Fig. 5 An example of lists used for plotting points and determining if those points are inside or outside the circle



Fig. 6 An alternative random number command to use for a circle with radius 10 units



Fig. 7 Solutions to part of the activity sheet

 $a^2 + b^2 > 1$  will be outside the circle. Determining  $a^2 + b^2$  can be computed easily since the values are already in List 1 and List 2. By setting up List 3 as

$$\sqrt{L1^2 + L2^2}$$

students can easily see where points will fall in relation to the circle. To be really accurate, students can look at each individual numerical value of L3 by scrolling down to "List" and looking at the bottom of the calculator screen (**fig. 5**). This value is much more accurate than the table value. It helps students see if the points are actually on the circle or inside or outside.

#### Conclusions

WHEN WE BEGAN USING THIS ACTIVITY, WE were excited to incorporate the calculator's random number generator with graphing and probability. As questions have been posed that add more technology and mathematics, we have become more excited every time we do the activity with a group of students or teachers. With the theme of "Connections" being so prominent in both the NCTM Standards and many state standards, this activity not only addresses a variety of mathematical topics but also connects them to the one central problem: constructing a model for a game of darts. In fact, the next time we use this activity with a group of students, we plan to ask them to list every mathematics topic they can think of that was used.

Another question that we have been asked while playing darts is the design of the dart board. One teacher noted that although her students needed practice graphing, they needed to work with integers rather than decimals. The calculator command for generating random numbers can easily be changed to generate ordered pairs from -10 to 10 instead of -1 to 1 (see **fig. 6**). The command shown will give values between -10 and 10, but to obtain integer values, students must set the mode to zero decimal places rather than two.

Most of the time we use a handout as an organizational tool to help students keep track of where they are going, what they are doing, and what they have done so far. **Hitting the Bull's-Eye Activity Sheet** is the handout used in our classes. It is simply a way for students to have a written record of the activity. We find it very useful for most students. **Figure 7** is a sample of one student's work.

Our enthusiasm about this activity comes from the numerous mathematical topics we are able to use and the technology we are able to incorporate at an appropriate level. Students love it. Teachers love it. We hope you love it as much as we do.  $\Box$ 

# **Hitting the Bull's-Eye Activity Sheet**

NAME \_\_\_\_\_

The dart board below has a radius 1 unit long. You are good enough so that all your darts land in the square region surrounding the dartboard but not good enough to hit the target every time. Use the calculator to simulate throwing darts at the dart board.



## How to generate random ordered pairs

The bull's-eye circle has a radius of 1. How can you obtain ordered pairs between -1 and 1?

## Procedure

Set your calculator mode at two decimal places to make the graphing easier. As you generate random ordered pairs, plot them on the target. Keep track of how many land inside the circle and outside the circle.

## Tally (for 25 darts)

Number of darts that land inside the circle: \_\_\_\_\_

Number of darts that land outside the circle: \_\_\_\_\_

# Your experimental probability

*P*(darts land inside the circle) = \_\_\_\_\_

## Another approach

Instead of plotting the points yourself, you can have the calculator generate the numbers and plot them on a unit circle.

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# **Hitting the Bull's-Eye Activity Sheet**

NAME \_\_\_\_\_

(continued)

## Procedure

Generate a list of random numbers from -1 to 1 and store those in L1. These numbers will serve as the *x*-coordinate of the ordered pair. The command needed for this is \_\_\_\_\_.

Generate another list of random numbers from -1 to 1 and store those in L2. These numbers will serve as the *y*-coordinate of the ordered pair. The command needed for this is \_\_\_\_\_.

To obtain a circle on the graph screen, you need  $y_1 = \_$  and  $y_2 = \_$ .

What do you need to obtain the scatter plot of points in L1 and L2?

How many points were inside the circle? \_\_\_\_\_

How many points were outside the circle?

*P*(darts land inside the circle) = \_\_\_\_\_

What is the theoretical probability that a dart will land inside the circle?

# Want to be more certain?

If you would like to know for sure whether your calculator points are inside or outside the circle, what can you do?

Radius of the circle = \_\_\_\_\_

The Pythagorean theorem says \_\_\_\_\_

Points inside the circle will be less than \_\_\_\_\_.

Points outside the circle will be greater than \_\_\_\_\_.

Command for L3: \_\_\_\_\_



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