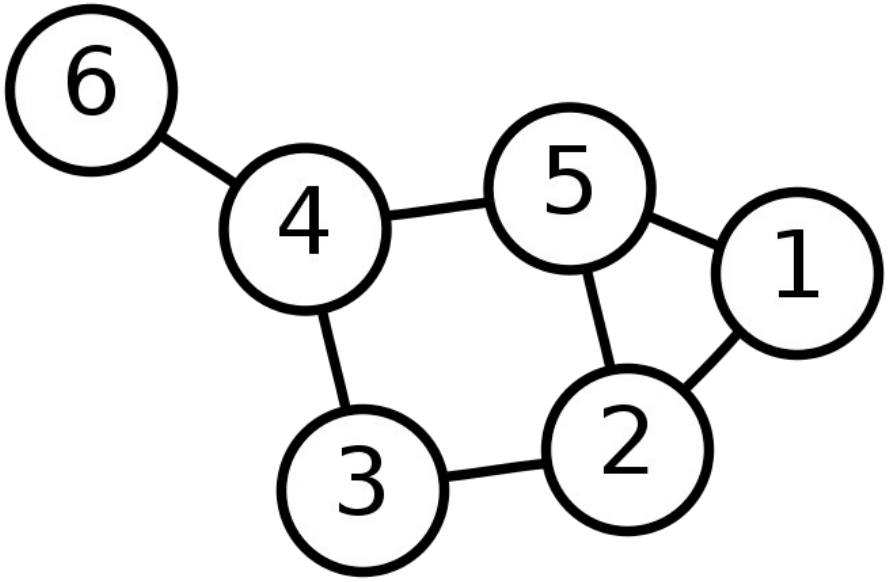
Graph Algorithms

- Recall basic Graph definition:
  - A graph is a pair (of sets) G = (V,E)
    - V is a finite set, called the “vertices”
    - E is a set of pairs of elements from V, denoting the “edges” of the graph



Review of types of graphs

- Weighted: comes with a function w: E → R
  - Denotes weight on each edge
- Simple: no self-loop edges
- Directed: edges are one-way
- Acyclic: no cycles
- Complete: all possible edges
- Connected: path between every pair of vertices

Important Graph Problems

- Minimal spanning tree
  - (in a weighted graph): find the smallest spanning tree, in terms of total edge weight
- Calculate shortest paths between nodes
  - Either weighted or unweighted
- Maximal independent set
  - Largest set of vertices that have no common edges

Prim's Algorithm

- Idea: start from a single node, then iteratively and "greedily" add vertices to the MST until all are added

```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Parallelizing Prim's Alg.

- Distribute columns or rows of the adjacency matrix
- The inner-loop can be re-framed as a reduce operation, using the MAX operation
- The structure Q can be a priority queue for efficiency, also distributed.
  - Then the selection of the min can be a reduction as well
  - Can keep a copy of the full queue on each node for easier bookkeeping

Parallel Prim's Algorithm

- Cost to select the minimum entry
  - O(n/p): scan n/p local part of vector on processors
  - O(log p) all-to-one reduction across processors
- Broadcast next node selected for membership
  - O(log p)
- Cost of locally updating d vector
  - O(n/p): replace d with min of d and matrix row
- Parallel time per iteration
  - O(n/p + log p)
- Total parallel time
  - O(n2/p + n log p)

Single-Start Shortest Path

- Use BFS!
  - The “d” property after completion is the length of the s.p.
  - The “pi” attribute is the predecessor in that s.p.
- Not very easy to make parallel :(

```
BFS(G, s)
 1  for each vertex u ∈ G.V − {s}
 2      u.color = WHITE
 3      u.d = ∞
 4      u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q ≠ ∅
11      u = DEQUEUE(Q)
12      for each v ∈ G.Adj[u]
13          if v.color == WHITE
14              v.color = GRAY
15              v.d = u.d + 1
16              v.π = u
17              ENQUEUE(Q, v)
18      u.color = BLACK
```

All-Pairs Shortest Path

- Do them all at once!
- Floyd-Warshall Algorithm:
  - Use dynamic programming
  - The length of the min path between nodes has a “nice” recursive structure
  - Idea: iterate k = 0 to |V| times, each time increasing the length of all paths

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$

Floyd-Warshall Algorithm

```
FLOYD-WARSHALL(W)
1  n = W.rows
2  D^(0) = W
3  for k = 1 to n
4      let D^(k) = (d_ij^(k)) be a new n × n matrix
5      for i = 1 to n
6          for j = 1 to n
7              d_ij^(k) = min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
8  return D^(n)
```

- We can “save” the paths in a matrix as we calculate the distance updates:

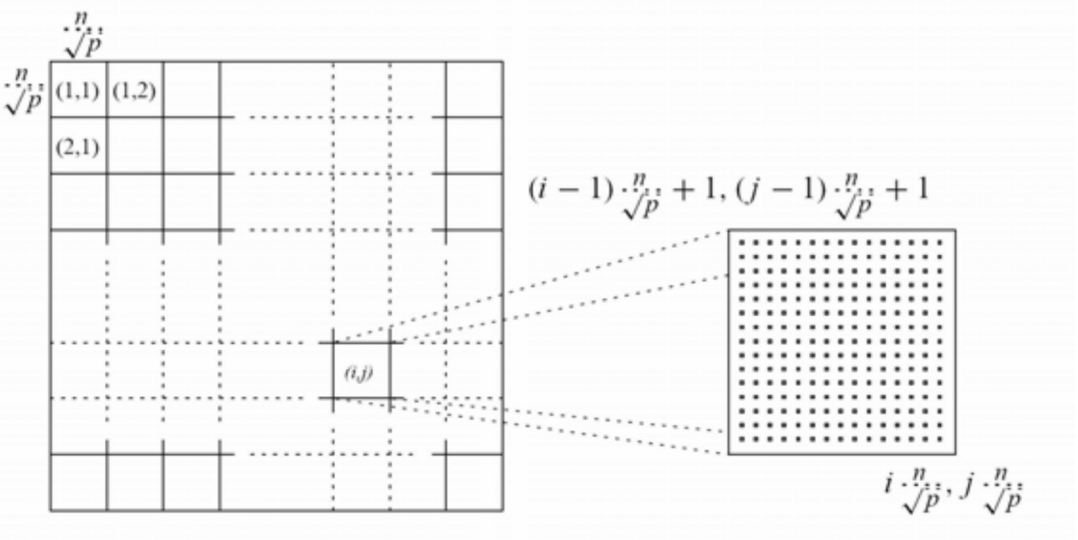
$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}. \end{cases}$$

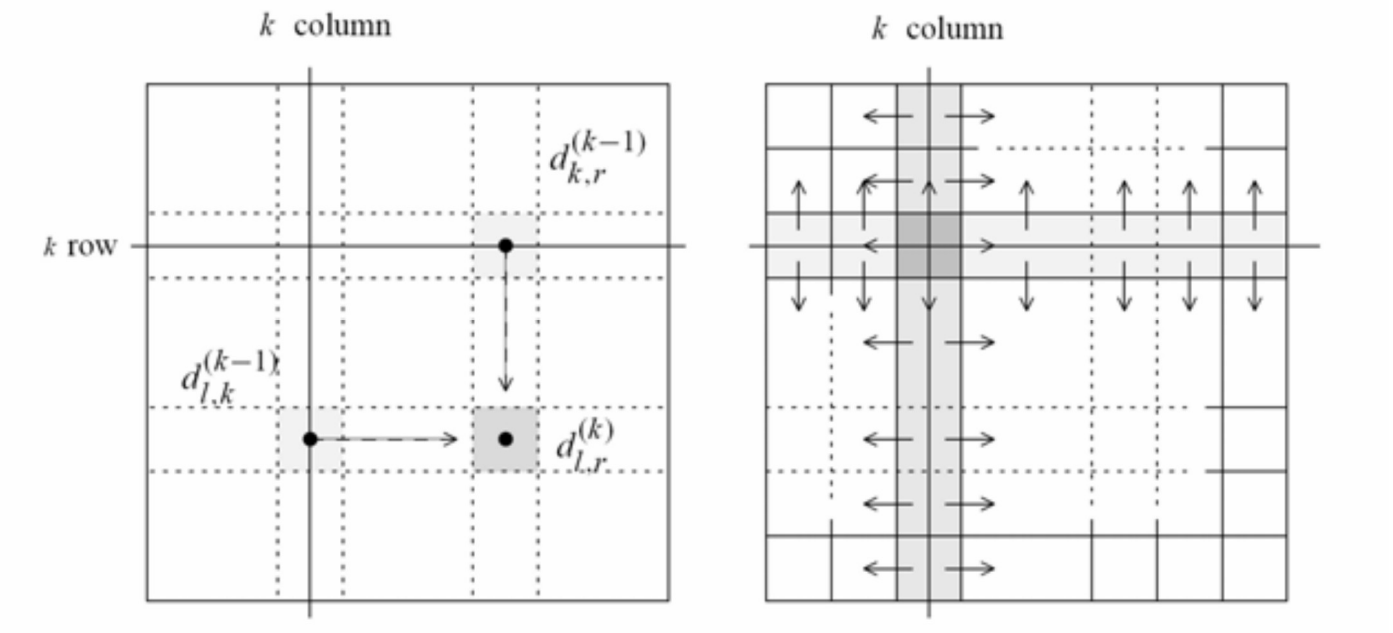
Parallel Floyd-Warshall

- Split the Adjacency matrix block-wise and parallelize the two inner loops:

```
procedure FLOYD_2DBLOCK(D^(0))
begin
    for k := 1 to n do
    begin
        each process P_{i,j} that has a segment of the k^th row of D^(k-1);
            broadcasts it to the P_{*,j} processes;
        each process P_{i,j} that has a segment of the k^th column of D^(k-1);
            broadcasts it to the P_{i,*} processes;
        each process waits to receive the needed segments;
        each process P_{i,j} computes its part of the D^(k) matrix;
    end
end FLOYD_2DBLOCK
```

Parallel Floyd-Warshall





Independent Sets

- Want to find the largest independent set of vertices (no edges between)
- Simple naive algorithm:
  - Start with an empty set of vertices
  - Add vertex with smallest degree; remove its neighbors from G
    - Repeat this until G has no vertices left
- Very difficult to parallize!
  - As in, can’t be done :(

Luby’s Max Ind. Set

- Randomized!
- Algorithm:
  - Start with an empty set
  - Assign random numbers to each vertex
  - Add vertices that got a number smaller than all neighbors to the set, remove nbrs
  - Repeat above two steps until G empty
- Good to parallelize?
  - Yes!

Strategies for Graphs

- Spoiler: use linear algebra!
  - Easy to distribute
  - Lets us leverage data reuse and cache locality
- Use the adjacency matrix of the graph. E.g:
  - If we have vector of distances from a single vertex (e.g. after BFS), we can re-formulate the kinds of loops above
  - Define a custom “inner product”:

$$y^t = x^t G \equiv \forall_i : y_j = \min\{x_j, \min_{i:\, G_{ij} \neq 0} \{x_i + 1\}\}$$

Example: Bellman-Ford

- Finds shortest path from a single source, allowing negative edge weights

Let s be given, and set $d(s) = 0$
Set $d(v) = \infty$ for all other nodes v
for $|E| - 1$ *times* **do**
  for *all edges* $e = (u, v)$ **do**
    Relax: **if** $d(u) + w_{uv} < d(v)$ **then**
      Set $d(v) \leftarrow d(u) + w_{uv}$

- Easily parallelized now, using the above vector-matrix product!

Example: Search Engine

- Model web-pages with hyperlinks between them as a graph:

$$L_{ij} = \begin{cases} 1 & \text{document } i \text{ points to document } j \\ 0 & \text{otherwise} \end{cases}$$

- Every webpage will have an “authority” and “hub” score
- If we say x is the vector of authorities and y the vector of hub scores:

$$x = L^t y$$
$$y = Lx$$

HITS (hypertext-induced search)

- Using the previous formulation (after some substitution), we get $x = LL^t x \text{ and } y = L^t L y$
- In other words, this is an *eigenvalue* problem!
  - Saved by linear algebra again!
- How do we compute these then?

Eigenvalue Problems

<preach>

Some of the most important types of problems in practical scenarios. Period.

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- Most of “data analysis” is really just solving fancy eigenvalue problems
- Eigenvalues capture “importance” of data, as just mentioned earlier in the context of web-pages

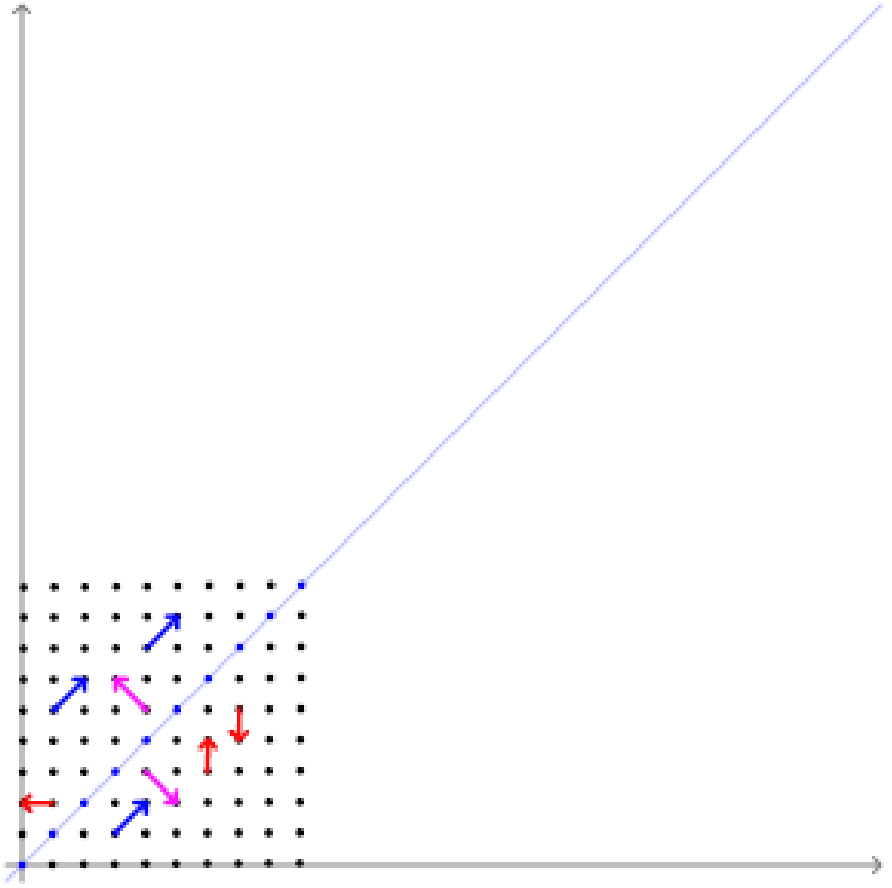
Eigenvalue Problems

- Eigenvalues/Eigenvectors:
  - Given a matrix A, we say that x is an eigenvector with corresponding (scalar) eigenvalue t if

$$Ax = tx$$

  - i.e. Applying A to x only *stretches* the vector
  - i.e. x lies along one of the important “directions” of the matrix A
- Matrices typically have many eigenvectors

Eigenvalues/Eigenvectors



Solving Eigenvalue Systems

- Given a matrix A
- Choose a vector (hopefully smartly) x_0
- Perform the sequential update: $x_i = Ax_{i-1}$
- This is called the power method because we end up with $x_i = A^i x_0.$
- If we got lucky with starting point, we would simply have $Ax_0 = \lambda x_0$ and $x_i = \lambda^i x_0.$

Example

- Try the following matrix A and starting vector

$$A = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}, \qquad x = (0, \ldots, 0, 1)^t.$$

- Try by hand for small examples. What happens?

Back to HITS...

Input: Adjacency matrix A of size $n \times m$ and number of iterations
Output: Authority and hub score vectors $\mathbf{x}$ and $\mathbf{y}$ respectively
$\mathbf{x} = (1, 1, \ldots, 1) \in \mathcal{R}^m$; $\mathbf{y} = (1, 1, \ldots, 1) \in \mathcal{R}^n$;
while *Iterations still left* **do** // Can detect when x & y don't change anymore
  for i=1,2,...,m **do**
    $x_j = \sum_{a_{ij}=1} y_i$; // This is a vector-matrix product!
  end
  for j=1,2,...,n **do**
    $y_i = \sum_{a_{ij}=1} x_j$; // This is a vector-matrix product!
  end
  Normalize($\mathbf{x}$); Normalize($\mathbf{y}$);
end

- Note: normalization means adjust to have unit length (root of sum of squares)

PageRank

- Start with vector p = e = vector of all 1's
- Use the adjacency matrix M where M_{ij} = 1 if page j has a link to page I
  - Then normalize it so that columns have a sum of 1, making it a stochastic matrix
- Choose small constant s > 0 to represent the chance of moving to a page that is not linked
- Iterate the process:
  - p = s*M*p + s * e

PageRank

- Assuming the above process converges, what can we say analytically about the situation?
- Fixed point is: $p = s*M*p + s*e$
- Then $(I_n - sM)\ p = s*e$
- So what we really need is $(I_n - sM)$ to have an inverse
- But if it does, it must be of the form
  - $I_n + sM + s^2M^2 + s^2M^2 \ldots$
  - Which does converge!

PageRank

- Using the above, we can also compute the inverse of (I_n- sM) and multiply s*e by it
- This gives a way to compute pagerank by a much simpler series of matrix-vector multiplications
- Recall that if we disallow “teleportation” by setting s = 0, the calculating the pagerank is the power iteration:
  - $p^k = M*p^{k-1}$
  - Which is (generally) a sparse product!