Extra Problem:

We proved PIE\(_n\) by induction but we could have proven it directly. Consider an arbitrary element \(x \in U\) and a collection \(\{A_1, A_2, \ldots, A_n\}\) of sets in \(U\). Then \(x\) is an element of \(k\) of the sets for some \(k\) such that \(0 \leq k \leq n\). We claim that:

\[
|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| + |A_1 \cap A_2 \cap A_3| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n| + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|
\]

Let us consider the number of times that \(x\) is counted in the sum on the right hand side. If \(k=0\) then \(x\) is in none of the sets so it is counted no times. If \(k>0\) then \(x\) is counted \(\binom{k}{1}\) times then subtracted \(\binom{k}{2}\) times added back in \(\binom{k}{3}\) etc. (Why?)

So in total \(x\) is counted \(\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \ldots + (-1)^{k-1} \binom{k}{k}\) times. If we can show that this sum is one for all \(k>0\) then PIE\(_n\) follows (why?)

1. Address the why?’s in the discussion above.

2. Prove that \(\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} = 1\) by showing that if \(E\) is the set of all non-empty subsets of a set \(A\) with an even number of elements and \(O\) is the number of subsets of a set \(A\) with an odd number of elements then \(|E| + 1 = |O|\). (Hint, consider a particular element \(x\) in \(A\) and pair each subset containing \(x\) with the same subset with \(x\) removed. Does this set up a 1-1 correspondence between \(O\) and \(E \cup \{\phi\}\)?)