Thm: If S and T are denumerable sets then SUT is a denumerable set.

Proof:
First, suppose that S and T are disjoint.

**Lemma 1:** If S and T are disjoint denumerable sets then SUT is a denumerable set.

Since S and T are denumerable there exists bijections \( f: \mathbb{N} \rightarrow S \) and \( g: \mathbb{N} \rightarrow T \).
Let \( h \) be defined as follows:

\[
    h(n) = \begin{cases} 
    f(n/2) & \text{if } n \text{ is even} \\
    g((n+1)/2) & \text{if } n \text{ is odd}
    \end{cases}
\]

Then \( h: \mathbb{N} \rightarrow SUT \) is a bijection. To show that \( h \) is onto, let \( x \) be an element of SUT. Then either \( x \) is in S or \( x \) is in T. If \( x \) is in S then there exists \( n \) such that \( f(n) = x \) since \( f \) is a bijection into S. In this case \( h(2n) = x \). On the other hand, if \( x \) is in T then there exist \( n \) such that \( g(n) = x \) and \( h(2n-1) = g(((2n-1)+1)/2) = g(n) = x \). In either case there is an \( m \) in \( \mathbb{N} \) such that \( h(m) = x \) so \( h \) is onto. To see that \( h \) is one to one, suppose that \( h(n) = h(m) \) for some \( n \) and \( m \) in \( \mathbb{N} \). Either \( n \) and \( m \) are both even, both are odd or one is even and the other is odd.

If \( n \) and \( m \) are both even then \( h(n) = f(n/2) = h(m) = f(m/2) \); so \( f(m/2) = f(n/2) \). However, since \( f \) is one to one, this means that \( n/2 = m/2 \) and \( m = n \).

If, on the other hand \( n \) and \( m \) are both odd then \( h(n) = g((n+1)/2) = h(m) = g((m+1)/2) \); so \( g((n+1)/2) = g((m+1)/2) \). Since \( g \) is one to one this means that \( (n+1)/2 = (m+1)/2 \) which implies that \( m = n \).

The only other possibility is that one is odd and the other even. WLOG, assume \( n \) is even and \( m \) is odd. This means that \( h(n) = f(n/2) \) is in S and \( h(m) = g((m+1)/2) \) is in T. But we assumed that S and T were disjoint so this cannot happen.

Therefore whenever \( h(n) = h(m) \), \( n = m \) and \( h \) is one to one. There is thus a bijection \( h: \mathbb{N} \rightarrow SUT \) so SUT is denumerable by definition.

Thus we know that the union of any two disjoint denumerable sets is denumerable.

Lemma 2: If S is denumerable and T is finite and S and T are disjoint then SUT is denumerable.

Proof: Since S is denumerable there exists a bijection \( f: \mathbb{N} \rightarrow S \). Since T is finite, T has \( n \) elements for some natural number \( n \) and there is a bijection \( g: \mathbb{N}_n \rightarrow T \). Let \( h: \mathbb{N} \rightarrow SUT \) be defined as follows:

\[
    h(m) = \begin{cases} 
    g(m) & \text{if } m \leq n \\
    f(m-n) & \text{if } m > n
    \end{cases}
\]

If \( x \) is in SUT then either \( x \) is in S or \( x \) is in T. If \( x \) is in S then there exists a natural number \( r \) such that \( f(r) = x \), since \( f \) is onto. Then \( h(r+n) = f(r) = x \). If on the other hand, \( x \) is in T then there exists a number, \( r \) in \( \mathbb{N}_n \) such that \( g(r) = x \). Then \( h(r) = x \) as well. So in both cases there is a natural number \( m \) such that \( h(m) = x \) so \( h \) is onto.

The function \( h \) is one to one as well. Suppose \( h(r) = h(m) \) for some natural numbers \( r \) and \( m \). Either, \( r \) and \( m \) are both in \( \mathbb{N}_n \), neither is in \( \mathbb{N}_n \) or one is in \( \mathbb{N}_n \) and the other is not. If both are in \( \mathbb{N}_n \) then \( h(r) = g(r) = h(m) = g(m) \) and since \( g \) is one to one, \( r = m \). If neither is in \( \mathbb{N}_n \) then both are greater than \( n \) and \( h(r) = f(r-n) = h(m) = f(m-n) \) and since \( f \) is one to one, \( r-n = m-n \) or \( r = m \). If, on the other hand \( r \leq n \) and \( m > n \) then \( h(r) \) is in T and \( h(m) \) is in S, since S and T are disjoint it is not possible for \( h(r) \) to equal \( h(m) \) in this case. Thus whenever \( h(m) = h(r) \) then \( m = r \) so \( h \) is one to one.
Now, suppose S and T are denumerable (but not disjoint) sets. Let A = the intersection of S and T. Since T/A is contained in T then T/A is either finite or denumerable. Also S and T/A are disjoint and SUT=SU(T/A). If T/A is denumerable then SU(T/A) is denumerable by Lemma 1. If T/A is finite then SU(T/A) is denumerable by Lemma 2.

This final argument along with Lemma 1 show that if S and T are denumerable sets then SUT is denumerable.

QED