Fractals

Key Concepts Used Congruence, Similarity, Iteration, Transformation.

Introduction Fractal Geometry is an emerging discipline that became quite popular in the early 1980’s. Although certain examples existed much earlier (for example, the Von Koch snowflake dates back to 1904), the field really took off with the advent of inexpensive high speed computers. Objects such as coastlines, clouds, snowflakes, and many plants exhibit geometric patterns, but not ones that are easily described using classic Euclidean shapes (e.g. squares, circles, etc.). However, they can be better described by fractal geometry which has made the phrase “the fractal geometry of nature” common in the literature. Fractals are objects that often have intricate “self-similar” structure. Mathematical fractals can be constructed using iteration, in many cases repeating a geometric process over and over. In this activity you will be introduced to two of the simplest and most famous mathematical fractals.

The Von Koch Snowflake

As our first example, consider the following construction:

1) Begin with a line segment of unit length:

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2) Remove the middle third of the segment and then add two line segments of length one third as follows:

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This is called the first iteration.

Note: The two new segments and the segment that was removed form the sides of an equilateral triangle whose side length is one third the length of the original segment.

3) With the four new segments, repeat the process above. That is, remove the middle third of each line segment and add two new line segments so that the 3 segments (the two added and one removed) form an equilateral triangle.

Exercise i): Draw a picture of the second iteration.

Exercise ii): Now assume that the $n^{th}$ iteration has been drawn for some $n \geq 1$. Explain how to create the $n + 1^{st}$ iteration.

Exercise iii): At the $n^{th}$ iteration how many line segments are there? What is the length of each line segment? What is the length of the $n^{th}$ iteration?

The Von Koch Curve is the limit curve of the $n^{th}$ iteration as we let $n$ approach infinity. Suppose now that instead of iterating this construction starting with a line segment, we start with three segments forming the boundary of an equilateral triangle and apply the process to each of the sides. The limiting curve is then called the Von Koch Snowflake. What do you think the snowflake looks like? Can we actually construct it?

Exercise iv): Look at some of the iterations you constructed. Can the concept of congruence be applied to any of the constructions? Why or why not? Can the concept of similarity be applied? Why or why not?

Exercise v): Many students have difficulty distinguishing the concepts of congruence and similarity. Explain how the construction of the Von Koch Curve could be used to help students with these concepts.

Exercise vi): You may have noticed that transformations may also be used in the construction of the Von Koch Curve. Which transformations may be applied to produce each iteration?
The Sierpinski Gasket

Next consider the following construction:

1) Begin with an equilateral triangle of unit side length, including its interior:

2) Remove an equilateral triangle from the original triangle as follows:

This is called the first iteration. Note that three smaller equilateral triangles remain.

3) With the three new triangles, repeat the process above. That is, remove an equilateral from each of these triangles to produce the second iteration.

*Exercise i)*: Draw a picture of the second iteration.
Exercise ii): Now assume that the $n^{th}$ iteration has been drawn for some $n \geq 1$. Explain how to create the $n + 1^{st}$ iteration.

Exercise iii): At the $n^{th}$ iteration how many triangles are there? What is the area of the $n^{th}$ iteration?

The Sierpinski Gasket is the limit curve of the $n^{th}$ iteration as we let $n$ approach infinity.

Exercise iv): Look at some of the iterations you constructed. Explain how the concepts of congruence and similarity can be applied these constructions?

Exercise v): Explain how the construction of the Sierpinski Gasket could be used to help students with these concepts.

Exercise vi): You may have noticed that transformations may also be used in the construction of the Sierpinski Gasket. Which transformations may be applied to produce each iteration?