ON THE QUANTUM PARAMETER IN THE QUANTUM COHOMOLOGY OF A FAMILY OF ODD SYMPLECTIC PARTIAL FLAG VARIETIES

CONNOR BEAN, CALEB SHANK, AND RYAN M. SHIFLER

Abstract. We will consider a particular family of odd symplectic partial flag varieties denoted by IF. In the quantum cohomology ring QH*(IF), we will show that $q_1 q_2 \cdots q_m$ appears m times in the quantum product $\tau_{Div_i} \star \tau_{id}$ when expressed as a sum in terms of the Schubert basis.

1. Introduction

Let IF := IF$(1, 2, \cdots, m; 2n+1)$ denote the family of odd symplectic partial flag varieties under consideration. This is the parameterization of sequences $(V_1 \subset V_2 \subset \cdots \subset V_m)$, $\dim V_i = i$, of subspaces of $\mathbb{C}^{2n+1}$ that are isotropic with respect to a general skew-symmetric form. The variety IF contains Schubert varieties $\{X(\lambda) : \lambda \in W^{odd}\}$ where W^{odd} is defined in Section 2. See [Mih07] for more details on odd symplectic flag varieties.

The quantum cohomology of a smooth variety X is a graded algebra over $\mathbb{Z}[q]$, $q = (q_1, q_2, \cdots, q_t)$, with a $\mathbb{Z}[q]$-basis given by classes in the cohomology ring $H^*(X)$. Multiplication is given by

$$\sigma_i \star \sigma_j = \sum_{d \geqslant 0; \sigma_k \in H^*(X)} q^d c_{i,j}^{k,d} \sigma_k$$

where $c_{i,j}^{k,d} = \langle \sigma_i, \sigma_j, \sigma_k^\vee \rangle_d$ is the Gromov-Witten invariant. The degree of q_i is

$$\deg q_i = \int_{Div_i{}^\vee} c_1(T_X)$$

where Div_i is the ith divisor class and $c_1(T_X)$ is the first Chern class of the tangent bundle of X. The study of the quantum cohomology of flag varieties has made progress. For example, Buch and Mihalcea use the technique of curve neighborhoods in [BM15] to produce an equivariant quantum Chevalley formula for any homogeneous variety G/P. Limited progress has been made in the study of the quantum cohomology of non-homogenous varieties. For example, the odd symplectic Grassmannian is studied in [Pec13, MS19]. This manuscript studies a family of odd symplectic partial flag varieties which are non-homoogenous varieties.

The quantum cohomology ring (QH*(IF), $\star$) is a graded algebra over $\mathbb{Z}[q] = \mathbb{Z}[q_1, \cdots, q_m]$ where $\deg q_i = 2$ for $1 \leqslant i \leqslant m-1$ and $\deg q_m = 2(n-m)+3$. The ring has a Schubert basis given by $\{\tau_\lambda := [X(\lambda)] : \lambda \in W^{odd}\}$. Here we take τ_{id} to be the class of the Schubert point pt and τ_{Div_i} to be a divisor class where $1 \leqslant i \leqslant m$. The ring multiplication is given by $\tau_\lambda \star \tau_\mu = \sum_{\nu,d} c_{\lambda,\mu}^{\nu,d} q^d \tau_\nu$ where $c_{\lambda,\mu}^{\nu,d}$ is the degree d Gromov-Witten invariant of τ_λ, τ_μ, and the Poicaré dual of τ_ν. We are now ready to state our main result. A more precise statement is given as Theorem 4.8.

2010 *Mathematics Subject Classification.* Primary 14N35; Secondary 14N15, 14M15.

Theorem 1. *Consider the quantum cohomology ring* $\mathrm{QH}^*(\mathrm{IF})$*. Then* $q_1q_2\cdots q_m$ *appears* m *times in the product* $\tau_{Div_i} \star \tau_{id}$ *when expressed as a sum in terms of the Schubert basis given by* $\{\tau_\lambda : \lambda \in W^{odd}\}$*.*

Our strategy will be to use curve neighborhood calculations which we explain next. Let X be a Fano variety. Let $d \in H_2(X,\mathbb{Z})$ be an effective degree. Recall that the moduli space of genus 0, degree d stable maps with two marked points $\overline{\mathcal{M}}_{0,2}(X,d)$ is endowed with two evaluation maps $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{0,2}(X,d) \to X$, $i = 1,2$ which evaluate stable maps at the i-th marked point.

Definition 1.1. Let $\Omega \subset X$ be a closed subvariety. The *curve neighborhood* of Ω is the subscheme

$$\Gamma_d(\Omega) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}\Omega) \subset X$$

endowed with the reduced scheme structure.

The notion of curve neighborhoods is closely related to quantum cohomology. Let $X(\lambda) \subset \mathrm{IF}$ be a Schubert variety, and let $\Gamma_d(X(\lambda)) = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ be the decomposition of the curve neighborhood into irreducible components. By the divisor axiom, any component Γ_i of "expected dimension" will contribute to the quantum product $\tau_{Div_i} \star \tau_\lambda$ with $(\tau_{Div_i}, d)\cdot a_i \cdot q^d[\Gamma_i]$, where a_i is the degree of $\mathrm{ev}_2 : \mathrm{ev}_1^{-1}(X(\lambda)) \to \Gamma_d(X(\lambda))$ over the given component (see [KM94] and Lemma 4.7). Therefore the main task is to find the components Γ_i of $\Gamma_{(1^m)}(pt)$, where $(1^m) = (\overbrace{1,\cdots,1}^{m})$, that are of expected dimension. That is, the following equation is satisfied:

$$\mathrm{codim}\ X(Div_i) + \mathrm{codim}\ pt = \deg q_1q_2\cdots q_m + \mathrm{codim}\ \Gamma_i.$$

These components are stated precisely in Proposition 4.6.

Broader Context Any curve neighborhood of a Schubert variety in the homogeneous space G/P is shown to be irreducible in [BM15]. This limits the number of times that q^d appears for a particular $d \in H_2(G/P,\mathbb{Z})$ in quantum products of Schubert classes. Examples of curve neighborhoods having two irreducible components are given for the odd symplectic Grassmannian in [MS19, PS24]. In particular, in the quantum Chevalley formula for the odd symplectic Grassmannian, q^1 appears twice in the quantum product of the divisor class and the class of the point when expressed as a sum in terms of the Schubert basis. The main purpose of this manuscript is to give a specific example where q^d appears a specified number of times as stated in Theorem 1.

2. Preliminaries

There are many possible ways to index the Schubert varieties of isotropic flag manifolds. Here we recall an indexation using signed permutations. Consider the root system of type C_{n+1} with positive roots

$$R^+ = \{t_i \pm t_j \mid 1 \leqslant i < j \leqslant n+1\} \cup \{2t_i \mid 1 \leqslant i \leqslant n+1\}$$

and the subset of simple roots

$$\Delta = \{\alpha_i := t_i - t_{i+1} \mid 1 \leqslant i \leqslant n\} \cup \{\alpha_{n+1} := 2t_{n+1}\}.$$

The coroot of $t_i \pm t_j \in R^+$ is $(t_i \pm t_j)^\vee = t_i \pm t_j$ and the coroot of $2t_i \in R^+$ is $(2t_i)^\vee = t_i$. The associated Weyl group W is the hyperoctahedral group consisting of *signed permutations*, i.e. permutations w of the elements $\{1,\cdots,n+1,\overline{n+1},\cdots,\overline{1}\}$ satisfying $w(\bar{i}) = \overline{w(i)}$ for all $w \in W$. For $1 \leqslant i \leqslant n$ denote by s_i the simple reflection corresponding to the root $t_i - t_{i+1}$ and s_{n+1} the simple reflection of $2t_{n+1}$. In particular, if $1 \leqslant i \leqslant n$ then $s_i(i) = i+1$,

$s_i(i+1) = i$, and $s_i(j)$ is fixed for all other j. Also, $s_{n+1}(n+1) = \overline{n+1}$, $s_{n+1}(\overline{n+1}) = n+1$, and $s_{n+1}(j)$ is fixed for all other j.

Each subset $I := \{i_1 < \ldots < i_r\} \subset \{1, \ldots, n+1\}$ determines a parabolic subgroup $P := P_I \subset \mathrm{Sp}_{2n+2}$ with Weyl group $W_P = \langle s_i \mid i \neq i_j \rangle$ generated by reflections with indices *not* in I. Let $\Delta_P := \{\alpha_{i_s} \mid i_s \notin \{i_1, \ldots, i_r\}\}$ and $R_P^+ := \mathrm{Span}_{\mathbb{Z}} \Delta_P \cap R^+$; these are the positive roots of P. Let $\ell\colon W \to \mathbb{N}$ be the length function and denote by W^P the set of minimal length representatives of the cosets in W/W_P. The length function descends to W/W_P by $\ell(uW_P) = \ell(u')$ where $u' \in W^P$ is the minimal length representative for the coset uW_P. We have a natural ordering $1 < 2 < \cdots < n+1 < \overline{n+1} < \cdots < \overline{1}$, which is consistent with our earlier notation $\overline{i} := 2n+3-i$.

Let P be the parabolic obtained by excluding the reflections $s_1, s_2, \cdots s_m$. Then the minimal length representatives W^P have the form $(w(1)|w(2)|w(3)|\cdots|w(m) < w(m+1) < \cdots < w(n) \leqslant n+1)$. Since the last $n+1-m$ labels are determined from the first m labels, we will identify an element in W^P with $(w(1)|w(2)|\cdots|w(m))$. Define $W^{odd} = \{w \in W^P : w(i) < \overline{1} \text{ for } 1 \leqslant i \leqslant m\}$.

Let $X^{ev} := \mathrm{IF}(1, 2, \cdots, m; 2n+2)$ be the symplectic partial flag that parameterizes sequences $(V_1 \subset V_2 \subset \cdots \subset V_m)$, $\dim V_i = i$, of subspaces of $\mathbb{C}^{2n+2}$ that are isotropic with respect to a skew-symmetric form. Here $P \subset \mathrm{Sp}_{2n+2}$ is the maximal parabolic subgroup corresponding to $I = \{1 < 2 < \cdots < m\}$ and $T_{2n+2} = (t_1, \cdots, t_{n+1}, t_{n+1}^{-1}, \cdots, t_1^{-1})$ is a maximal torus for X^{ev}. The Schubert varieties of X^{ev} are indexed by $\lambda \in W^P$ and written as $X(\lambda)$. Since IF is identified with the Schubert variety $X(\overline{2}\overline{3}\cdots\overline{m}\overline{m+1}) \subset X^{ev}$, the Schubert varieties of IF are $\{X(\lambda) : \lambda \in W^{odd}\}$. In addition IF is smooth. The quantum cohomology ring $\mathrm{QH}^*(\mathrm{IF})$ has a Schubert basis given by $\{\tau_\lambda := [X(\lambda)] : \lambda \in W^{odd}\}$. We have that $T = (t_1, \cdots, t_{n+1}, t_{n+1}^{-1}, \cdots, t_2^{-1})$ is a maximal torus for IF and $\dim \mathrm{IF} = m(2n-m+1)$. Next we will give notation to state the Bruhat order.

Example 2.1. Consider $\mathrm{IF}(1, 2, 3; 11)$. This identifies with the Schubert variety $X(\overline{2}\overline{3}\overline{4})$ in $\mathrm{IF}(1, 2, 3; 12)$. Here $(1|\overline{2}|3), (5|\overline{4}|2) \in W^{odd}$ while $(3|\overline{1}|2) \notin W^{odd}$.

Definition 2.2. Let $\lambda, \delta \in W^{odd}$ where $\lambda_i = \lambda(i)$ and $\delta_i = \delta(i)$. Then define the following:

(1) $\Lambda^k := \langle \Lambda_1^k < \Lambda_2^k < \cdots < \Lambda_k^k \rangle$ where $\{\Lambda_1^k, \Lambda_2^k, \cdots, \Lambda_k^k\} = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$;
(2) $\Delta^k := \langle \Delta_1^k < \Delta_2^k < \cdots < \Delta_k^k \rangle$ where $\{\Delta_1^k, \Delta_2^k, \cdots, \Delta_k^k\} = \{\delta_1, \delta_2, \cdots, \delta_k\}$;
(3) $\Lambda^k \leqslant \Delta^k$ if $\Lambda_i^k \leqslant \Delta_i^k$ for all $1 \leqslant i \leqslant k$.

Lemma 2.3 (Bruhat Order [Pro82])**.** *Let $\lambda, \delta \in W^P$. Then $\lambda \leqslant \delta$ if and only if $\Lambda^k \leqslant \Delta^k$ for all $1 \leqslant k \leqslant m$. In particular, if $\lambda, \delta \in W^{odd}$ then $X(\lambda) \subset X(\delta)$ if and only if $\lambda \leqslant \delta$.*

3. The Moment Graph

Sometimes called the GKM graph, the *moment graph* of a variety with an action of a torus T has a vertex for each T-fixed point, and an edge for each 1-dimensional torus orbit. The description of the moment graphs for flag manifolds is well known, and it can be found in [Kum02, Ch. XII]. In this section we consider the moment graphs for IF and X^{ev}.

Definition 3.1. The moment graph of X^{ev} has a vertex for each $w \in W^P$, and an edge $w \to ws_\alpha$ for each

$$\alpha \in R^+ \backslash R_P^+ = \{t_i - t_j \mid 1 \leqslant i \leqslant m, i < j \leqslant m+1\} \cup \{t_i + t_j, 2t_i \mid 1 \leqslant i \leqslant m, 1 \leqslant i < j \leqslant m+1\}.$$

This edge has degree $d = (d_1, d_2, \cdots, d_m)$, where $\alpha^\vee + \Delta_P^\vee = d_1\alpha_1^\vee + d_2\alpha_2^\vee + \cdots + d_m\alpha_m^\vee + \Delta_P^\vee$. We will say that $d = (d_1, d_2, \cdots, d_m) \leqslant d' = (d'_1, d'_2, \cdots, d'_m)$ if $d_i \leqslant d'_i$ for all $1 \leqslant i \leqslant m$.

Definition 3.2. The moment graph of IF is the full subgraph of X^{ev} determined by the vertices $w \in W^{odd}$.

Next we classify the positive roots by their degree.

Definition 3.3. Let $(0^a 1^b 2^c) := (\overbrace{0, \cdots, 0}^{a}, \overbrace{1, \cdots, 1}^{b}, \overbrace{2, \cdots, 2}^{c})$. Define the following to describe moment graph combinatorics.

(1) Define the following sets which partitions $R^+ \backslash R_P^+$.
  (a) $R^+_{(0^{i-1}1^{j-i}0^{m-j+1})} = \{t_i - t_j : 1 \leqslant i < j \leqslant m\}$;
  (b) $R^+_{(0^{i-1}1^{m-i+1})} = \{t_i \pm t_j : 1 \leqslant i \leqslant j, m < j \leqslant n+1\} \cup \{2t_i : 1 \leqslant i \leqslant m\}$;
  (c) $R^+_{(0^{i-1}1^{j-i}2^{m-j+1})} = \{t_i + t_j : 1 \leqslant i < j \leqslant m\}$.

(2) A *chain of degree* d is a path in the (unoriented) moment graph where the sum of edge degrees equals d. We will use the notation $uW_P \xrightarrow{d} vW_P$ to denote such a path.

In the next lemma we give a formula for the degree d of a chain which is useful to calculate curve neighborhoods. In particular, we will see that the degree of a chain is determined by summing the weights of the edges included in the chain (repetitions are allowed) in the moment graph.

Lemma 3.4 ([FW04], Page 8)**.** *Let $u, v \in W^P$ be connected by a degree d chain*

$$(uW_P \xrightarrow{d} vW_P) = (uW_P \to us_{\alpha_1}W_P \to \cdots \to us_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_t}W_P)$$

where $vW_P = us_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_t}W_P$ and the α_j are in $R^+ \backslash R_P^+$. Then $d = (O_1 + D_1, O_2 + D_2, \cdots, O_m + D_m)$ where

$$O_i = \sum_{\substack{a \leqslant i-1 \\ a+b \geqslant i}} \# \left\{ \alpha_j \in R^+_{(0^a 1^b 2^{m-a-b})} \right\} \text{ and } D_i = 2 \cdot \sum_{a+b \leqslant i-1} \# \left\{ \alpha_j \in R^+_{(0^a 1^b 2^{m-a-b})} \right\}.$$

4. Proof of main result

We begin this section by stating Proposition 4.1 which gives curve neighborhoods, defined in Definition 1.1, a combinatorial interpretation in terms of the moment graph. Then Lemmas 4.2 and 4.3 demonstrate that $\lambda \in W^{odd}$ is constrained when it is reached by a chain of degree less than or equal to (1^m). This follows with Lemmas 4.4 and 4.5 which gives a precise statement of $\Gamma_{(1^m)}(pt)$ in Proposition 4.6. Finally, we present our main result in Theorem 4.8 which follows from Lemma 4.7.

Proposition 4.1 ([BM15])**.** *Let $\lambda \in W^{odd}$. In the moment graph of* IF*, let $\{v^1, \cdots, v^s\}$ be the maximal vertices (for the Bruhat order) which can be reached from any $u \leqslant \lambda$ using a chain of degree d or less. Then $\Gamma_d(X(\lambda)) = X(v^1) \cup \cdots \cup X(v^s)$.*

Proof. Let $Z_{\lambda,d} = X(v^1) \cup \cdots \cup X(v^s)$. Let $v := v^i \in Z_{\lambda,d}$ be one of the maximal T-fixed points. By the definition of v and the moment graph there exists a chain of T-stable rational curves of degree less than or equal to d joining $u \leqslant \lambda$ to v. It follows that there exists a degree d stable map joining $u \leqslant \lambda$ to v. Therefore $v \in \Gamma_d(X(\lambda))$, thus $X(v) \subset \Gamma_d(X(\lambda))$, and finally $Z_{\lambda,d} \subset \Gamma_d(X(\lambda))$.

For the converse inclusion, let $v \in \Gamma_d(X(\lambda))$ be a T-fixed point. By [MM18, Lemma 5.3] there exists a T-stable curve joining a fixed point $u \in X(\lambda)$ to v. This curve corresponds to a path of degree d or less from some $u \leqslant \lambda$ to v in the moment graph of $\mathrm{IG}(k, 2n+1)$. By maximality of the v^i it follows that $v \leqslant v^i$ for some i, hence $v \in X(v^i) \subset Z_{\lambda,d}$, which completes the proof. ∎

Lemma 4.2. *Let $\mathcal{C} : idW \xrightarrow{d} \lambda W$ be a chain in the moment graph of* IF *where $d \leqslant (1^m)$. Then we have the inequality*

$$\left|\Lambda^k \cap \{1, 2, \cdots, k\}\right| \geqslant k - 1.$$

Proof. Suppose $\left|\Lambda^k \bigcap \{1, 2, \cdots, k\}\right| < k - 1$. Then there are at at least two elements $\Lambda^k_{a_1}, \Lambda^k_{b_1} \in \Lambda^k$ such that $\Lambda^k_{a_1}, \Lambda^k_{b_1} > k$. Since $\Lambda^k_{a_1} > k$ there exists a reflection in the chain $\mathcal{C}$ corresponding to $t_{a_1} - t_{a_2}$ where $a_1 \leqslant k$ and $a_2 > k$. Also, since $\Lambda^k_{b_1} > k$ there exists a reflection in the chain $\mathcal{C}$ corresponding to $t_{b_1} - t_{b_2}$ where $b_1 \leqslant k$ and $b_2 > k$. Therefore, $d_k \geqslant 2$. But $d_k \leqslant 1$. The result follows. □

Lemma 4.3. *Let $\mathcal{C} : idW \xrightarrow{d} \lambda W$ be a chain in the moment graph of* IF *where $d \leqslant (1^m)$ and $\bar{j} \in \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ for some $2 \leqslant j \leqslant m$. The chain $\mathcal{C}$ has a reflection corresponding to the root $2t_j$. In particular, $1 \in \Lambda^j$.*

Proof. Consider the chain $\mathcal{C} : idW_P \xrightarrow{d} \lambda W_P$. One of the following three cases must have occurred.

1. The chain $\mathcal{C}$ has a reflection corresponding to the root $2t_j$;
2. The chain $\mathcal{C}$ has two reflections corresponding to two roots of the form $t_a \pm t_b$ where $a \leqslant m$ and $b \geqslant m$;
3. The chain $\mathcal{C}$ has a reflection corresponding to the root $t_a + t_b$ where $a, b \leqslant m$ and $a < b$.

In the first case we have that

$$(2t_j)^\vee = t_j = (t_j - t_{j+1}) + (t_{j+1} - t_{j+2}) + \cdots + (t_{n-1} - 1t_n) + t_n.$$

In particular, $d_i \leqslant 1$ for all $1 \leqslant i \leqslant m$. In the second case, the coefficient of $t_m - t_{m+1}$ is 1 when $t_a \pm t_b$ and $t_c \pm t_d$ ($a, c \leqslant m$ and $b, d \geqslant m$), are written as a sum of simple roots. Thus, $d_m \geqslant 2$. This is not possible. In the third case, the coefficient of $t_m - t_{m+1}$ is 2 when $t_a + t_b$ ($a, b \leqslant m$ and $a < b$) is written as a sum of simple roots. This is not possible. Therefore, the chain $\mathcal{C}$ has a reflection corresponding to the root $2t_j$. Finally, if $1 \notin \Lambda^j$, then $d_j \geqslant 2$ or $\bar{1}$ appears in λ. Neither is possible. This completes the proof. □

Lemma 4.4. *Let $\mathcal{C} : idW \xrightarrow{d} \lambda W$ be a chain in the moment graph of* IF *such that $d \leqslant (1^m)$.*

(1) If $\Lambda^m_m \leqslant \overline{m+1}$ then $X(\lambda) \subset X(\overline{m+1}|2|3|\cdots|m)$.
(2) If $\bar{j} \in \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$, where $2 \leqslant j \leqslant m$, then

$$X(\lambda) \subset X(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m).$$

Proof. We will prove Part (1) first. Let $1 \leqslant k \leqslant m$, $\delta = (\overline{m+1}|2|3|\cdots|m)$, and $\Lambda^m_m \leqslant \overline{m+1}$. It follows that $\Delta^k = (2 < 3 < \cdots < k < \overline{m+1})$. Also, $\left|\Lambda^k \cap \{1, 2, \cdots, k\}\right| \in \{k-1, k\}$ by Lemma 4.2. If $\left|\Lambda^k \cap \{1, 2, \cdots, k\}\right| = k$ then clearly $\Lambda^k \leqslant \Delta^k$.

Suppose that $\left|\Lambda^k \cap \{1, 2, \cdots, k\}\right| = k - 1$. Then $\Lambda^k = (1 < 2 < \cdots < \hat{i} < \cdots < k < \lambda_j)$ where i is removed and $\lambda_j \leqslant \Lambda^m_m \leqslant \overline{m+1}$. It follows that $\Lambda^k \leqslant \Delta^k$. Therefore, $\lambda \leqslant \delta$ and Part (1) follows by Lemma 2.3.

Next we will prove Part (2). Let $1 \leqslant k \leqslant m$, $\bar{j} \in \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$, where $2 \leqslant j \leqslant m$, and $\delta = (\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m)$. There are two cases for Δ^k.

1. If $k \leqslant j - 1$ then $\Delta^k = (2 < 3 < \cdots < k < \bar{j})$;
2. if $k \geqslant j$ then $\Delta^k = (1 < 2 < 3 \cdots < j - 1 < j + 1 < \cdots < k < \bar{j})$.

If $\left|\Lambda^k \cap \{1, 2, \cdots, k\}\right| = k$ then clearly $\Lambda^k \leqslant \Delta^k$.

Suppose that $\left|\Lambda^k \cap \{1, 2, \cdots, k\}\right| = k - 1$. Then $\Lambda^k = (1 < 2 < \cdots < \hat{i} < \cdots < k < \bar{j})$ where i is removed. If $k \leqslant j - 1$ then clearly $\Lambda^k \leqslant \Delta^k$. If $k \geqslant j$ then 1 must be included in Λ^k by Lemma 4.3. So, if $k \geqslant j$, we have that $\Lambda^k \leqslant \Delta^k$. Therefore, $\lambda \leqslant \delta$ and Part (2) follows by Lemma 2.3. This concludes the proof. ☐

Lemma 4.5. *We have the following permutation length calculation*

$$\ell(\overline{m+1}|2|3|\cdots|m) = \ell(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m) = 2n$$

for $2 \leqslant j \leqslant m$. In particular, the union

$$X(\overline{m+1}|2|3|\cdots|m) \cup \left(\bigcup_{j=2}^{m} X(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m)\right)$$

has m irreducible components of dimension $2n$.

Proof. The lengths $\ell(\overline{m+1}|2|3|\cdots|m)$ and $\ell(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m)$ are calculated by counting the number of simple reflections in a reduced word of the given permutation. ☐

Proposition 4.6. *Let $n \in \mathbb{Z}^+$ and consider* IF*. Then $\Gamma_{(1^m)}(pt)$ has m irreducible components of dimension $2n$. Specifically,*

$$\Gamma_{(1^m)}(pt) = X(\overline{m+1}|2|3|\cdots|m) \cup \left(\bigcup_{j=2}^{m} X(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m)\right).$$

Proof. This is an immediate consequence of Proposition 4.1 and Lemmas 4.4 and 4.5. ☐

Lemma 4.7 (divisor axiom, [KM94])**.** *Let $I_d(\tau_\lambda, \tau_\delta, \tau_{Div_i})$ be the 3-point Gromov-Witten Invariant of τ_λ, τ_δ, and τ_{Div_i} and $I_d(\tau_\lambda, \tau_\delta)$ be the 2-point Gromov-Witten Invariant of τ_λ and τ_δ. Then the divisor axiom states*

$$I_d(\tau_\lambda, \tau_\delta, \tau_{Div_i}) = (\tau_{Div_i}, d) I_d(\tau_\lambda, \tau_\delta).$$

In particular, any component Γ_i of $\Gamma_d(X(\lambda)) = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ that satisfies

$$\textit{codim } X(Div_i) + \textit{codim } pt = \textit{deg } q^{(1^m)} + \textit{codim } \Gamma_i$$

will contribute to the quantum product $\tau_{Div_i} \star \tau_\lambda$ with $(\tau_{Div_i}, d) \cdot a_i \cdot q^d[\Gamma_i]$, where a_i is the degree of $\mathrm{ev}_2 : \mathrm{ev}_1^{-1}(X(\lambda)) \to \Gamma_d(X(\lambda))$ over the given component.

Theorem 4.8. *In the quantum cohomology ring* $\mathrm{QH}^*(\mathrm{IF})$ *we have that*

$$\tau_{Div_i} \star \tau_{id} = (\tau_{Div_i}, d) q_1 q_2 \cdots q_m \left(a_1 \tau_{(\overline{m+1}|2|3|\cdots|m)} + \sum_{j=2}^{m} a_j \tau_{(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m)} \right) + \textit{other terms}$$

where a_j is the degree of $\mathrm{ev}_2 : \mathrm{ev}_1^{-1}(pt) \to \Gamma_d(X(\lambda))$ over $X(\overline{m+1}|2|3|\cdots|m)$ when $j = 1$ and $X(\bar{j}|2|3|\cdots|j-1|1|j+1|\cdots|m)$ when $2 \leqslant j \leqslant m$.

Proof. First notice that each irreducible component of $\Gamma_{(1^m)}(id)$ is of expected dimension. That is, codim $X(Div_i)$ + codim $pt = \deg q^{(1^m)} + (\dim \mathrm{IF} - 2n)$. The result follows by the divisor axiom. ☐

References

[BM15] Anders Buch and Leonardo C. Mihalcea, *Curve neighborhoods of Schubert varieties*, Journal of Differential Geometry **99** (2015), no. 2, 255–283.

[FW04] Wiliam Fulton and Christopher Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. **13** (2004), no. 4, 641–661.

[KM94] Maxim Kontsevich and Yuri Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Communications in Mathematical Physics **164** (1994), no. 3, 525–562.

[Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Mih07] Ion Alexandru Mihai, *Odd symplectic flag manifolds*, Transformation Groups **12** (2007), no. 3, 573–599.

[MM18] Augustin-Liviu Mare and Leonardo C. Mihalcea, *An affine quantum cohomology ring for flag manifolds and the periodic Toda lattice*, Proceedings of the London Mathematical Society **116** (2018), no. 1, 135–181.

[MS19] Leonardo C. Mihalcea and Ryan M. Shifler, *Equivariant quantum cohomology of the odd symplectic Grassmannian*, Mathematische Zeitschrift **291** (2019), no. 3-4, 1569–1603.

[Pec13] Clélia Pech, *Quantum cohomology of the odd symplectic grassmannian of lines*, Journal of Algebra **375** (2013), 188–215.

[Pro82] Robert Proctor, *Classical Bruhat orders and lexicographic shellability*, Journal of Algebra **77** (1982), no. 1, 104–126.

[PS24] Clelia Pech and Ryan M. Shifler, *Curve neighborhoods of Schubert varieties in the odd symplectic Grassmannian*, Transformation Groups **29** (2024), no. 1, 361–408.

Department of Mathematical Sciences, Henson Science Hall, Salisbury University, Salisbury, MD 21801

Email address: cbean2@gulls.salisbury.edu

Email address: cshank3@gulls.salisbury.edu

Email address: rmshifler@salisbury.edu