MINIMUM QUANTUM DEGREES WITH MAYA DIAGRAMS IN A FAMILY OF ODD ORTHOGONAL PARTIAL FLAG VARIETIES

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Abstract. We refine the criterion of Fulton and Woodward for the smallest powers of the quantum parameter q that occur in a product of Schubert classes in the (small) quantum cohomology of the odd orthogonal partial flag variety OF := OF$(1,2;2n{+}1)$. Our approach uses Maya diagrams and yields a combinatorial proof that the minimal quantum degrees are unique for OF.

1. Introduction

The vector space $V \subset \mathbb{C}^{2n+1}$ is isotropic with respect to a non-degenerate symmetric bilinear form ω if $\omega(x,y)=0$ for all $x,y \in V$. Let $\mathrm{OF} := \mathrm{OF}(1,2;2n+1)$ denote the odd orthogonal partial flag given by

$$\mathrm{OF}(1,2;2n+1) = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^{2n+1} \mid \dim V_i = i \text{ and } V_i \text{ isotropic with respect to } \omega\}.$$

Let $\mathrm{QH}^*(\mathrm{OF})$ be the small quantum cohomology with Schubert classes σ_w, $w \in W^P$. The set W^P is the minimum length coset representative of the associated Weyl group W modded by a parabolic P that corresponds to the set I. The set W^P is defined in Section 2. We denote the Poincare dual of σ_v by σ^v or $\sigma_{v^\vee}$. The small quantum cohomology ring $\mathrm{QH}^*(\mathrm{OF})$ is a graded $\mathbb{Z}[q]$-module. Multiplication is given by

$$\sigma^v \star \sigma_w = \sum_{u,d \geqslant 0} c_{v^\vee,w}^{u,d} q^d \sigma_u$$

where $c_{v^\vee,w}^{u,d}$ is the Gromov-Witten invariant that enumerates the rational curves of degree d. Given any element $\tau \in \mathrm{QH}^*(\mathrm{OF})$, we say that q^d **occurs** in τ if the coefficient of $q^d\sigma_w$ is not zero for some $w \in W^P$.

In [Shi25] Maya diagrams were used for Type A (partial) flag varieties to refine a criterion by Fulton and Woodward in [FW04] to calculate the smallest powers of q that occur in the quantum product of two Schubert classes. The criterion is given as Proposition 4.1. The refinement both simplifies and reduces the number of cases that need to be checked to find the smallest quantum degrees. That is, a priori, it is not clear from the Fulton and Woodward criterion that the smallest quantum degrees are unique. Fulton and Woodward's criterion is for general homogeneous G/P. So, it is natural to pursue using Maya diagrams to refine their criterion in other types. This manuscript focuses on Type B odd orthogonal partial flag varieties OF. We are ready to state our main results from Section 8.

Theorem 1.1. *Let $v, w \in W^P$.*

1. *The minimum quantum degree that occurs in $\sigma^v \star \sigma_w$ is unique using combinatorial methods.*
2. *The minimum quantum degree can be found using an algorithm built on Maya diagrams that refines the criterion by Fulton and Woodward.*

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The results in this article are combinatorial in the sense that we use Maya diagrams to describe the chains in the moment graph that Fulton and Woodward defined in [FW04] (see Proposition 4.1). Like in the Type A case found in [Shi25] we introduce a generalized notion of rim hook removals on Maya diagrams which is found in Section 6. In addition, Maya diagrams give a characterization of the Bruhat order by slightly modifying a theorem by Proctor in [Pro82, Theorem 5BC] and are stated herein as Proposition 5.7.

In the present work, a new challenge is introduced. In the Type A case, it was enough to study how the Bruhat order behaved on the rows of the Maya diagrams to know which rim hook to use. That is not the case for OF and is highlighted in Theorem 8.5.

2. Preliminaries

The vector space $V \subset \mathbb{C}^{2n+1}$ is isotropic with respect to a non-degenerate symmetric bilinear form ω if $\omega(x, y) = 0$ for all $x, y \in V$. Let $\mathrm{OF} := \mathrm{OF}(1, 2; 2n + 1)$ denote the odd orthogonal partial flag given by

$$\mathrm{OF}(1,2;2n+1) = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^{2n+1} \mid \dim V_i = i \text{ and } V_i \text{ isotropic with respect to } \omega\}.$$

Here we give some notation that is useful for describing the geometry of $\mathrm{OF}(1, 2; E)$. Consider the root system of type B_n with positive roots $R^+ = \{t_i \pm t_j \mid 1 \leqslant i < j \leqslant n\} \cup \{t_i \mid 1 \leqslant i \leqslant n\}$ and the subset of simple roots $\Delta = \{\alpha_i := t_i - t_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{\alpha_{n+1} := t_{n+1}\}$. Let $(t_i \pm t_j)^\vee = (t_i \pm t_j)$ when $1 \leqslant i < j \leqslant n$ and $(t_i)^\vee = 2t_i$ when $1 \leqslant i \leqslant n$. The associated Weyl group W is the hyperoctahedral group consisting of *signed permutations*. That is, $W = \{w \in S_{2n} \mid w(\bar{i}) = \overline{w(i)}\}$ where S_{2n} is the group of permutations on $\{1, 2, \cdots, n, \bar{n}, \cdots, \bar{2}, \bar{1}\}$ and $\bar{i} = 2n + 1 - i$. Let $\alpha \in R^+$ and let s_α denote the corresponding reflection.

Example 2.1. Let $w \in W$. Then $w(s_{t_2+t_5}(2)) = w(\bar{5})$ and $w(s_{t_2+t_5}(5)) = w(\bar{2})$.

Let $\Delta_P = \{\alpha_i \mid i \notin \{1, 2\}\}$, $R_P^+ = \mathrm{Span}_{\mathbb{Z}} \Delta_P \cap R^+$, and $\Delta_P^\vee = \{\alpha^\vee : \alpha \in \Delta_P\}$. We also need the Weyl group $W_P = \langle s_{\alpha_i} \mid i \neq 1, 2\rangle$. Denote by W^P the set of minimal length representatives of the cosets in W/W_P. The minimal length representatives W^P have the form $(w(1)|w(2)|w(3) < \cdots < w(n) \leqslant n + 1)$. Since the last $n - 1$ labels are determined from the first 2 labels, we will identify an element in W^P with $(w(1)|w(2))$.

Example 2.2. We have $w = (1|\bar{3}) = (1|\bar{3}|2)$ and $w(\bar{2}) = \overline{w(2)} = \bar{\bar{3}} = 3$.

3. Moment Graph

We say that two unequal elements v and w in W^P are **adjacent** if there is a reflection $S_\alpha \in W$ such that $w = vs_\alpha$. Define $d(v, w) = (d_1, d_2)$ where

$$\alpha^\vee + \Delta_P^\vee = d_1\alpha_1^\vee + d_2\alpha_2^\vee + \Delta_P^\vee.$$

The moment graph has a vertex for each $w \in W^P$, and an edge $w \to ws_\alpha$ for each

$$\alpha \in R^+ \backslash R_P^+ = \{t_i - t_j \mid 1 \leqslant i \leqslant 2, i < j \leqslant n+1\} \cup \{t_i + t_j, 2t_i \mid 1 \leqslant i \leqslant 2, 1 \leqslant i < j \leqslant n+1\}.$$

Geometrically, this edge corresponds to a curve $C_\alpha(w)$ joining w and ws_α. The curve $C_\alpha(w)$ has degree $d = (d_1, d_2)$, where

$$\alpha^\vee + \Delta_P^\vee = d_1\alpha_1^\vee + d_2\alpha_2^\vee + \Delta_P^\vee.$$

Example 3.1. Consider $w = (1|3)$ and $\alpha = t_1 + t_3$. Then

$$\begin{aligned} \alpha^\vee + \Delta_P^\vee &= (t_1 + t_3) + \Delta_P^\vee \\ &= \overbrace{(t_1 - t_2)}^{\alpha_1^\vee} + \overbrace{(t_2 - t_3)}^{\alpha_2^\vee} + 2t_3 + \Delta_P^\vee \end{aligned}$$

Therefore, the curve $C_\alpha(w)$ has degree $d = (1, 1)$.

Define a **chain** $\mathcal{C}$ from v to w in W^P to be a sequence $u_0, u_1, \cdots, u_r$ in W^P such that u_{i-1} and u_i are adjacent for $1 \leqslant i \leqslant r$ and $u_0 \leqslant v$ and $w \leqslant u_r$. We say that the chain **originates** at u_0 and **terminates** at u_r. For any chain $u_0, u_1, \cdots, u_r$ we define the **degree** of the chain $\mathcal{C}$, denoted $\deg_{\mathcal{C}}(v, w)$, to be the sum of the degrees $d(u_{i-1}, u_i)$ for $1 \leqslant i \leqslant r$. Note that there is a chain of degree 0 between v and w exactly when $w \leqslant v$.

4. Quantum Cohomology

Let $\mathrm{QH}^*(\mathrm{OF})$ denote the quantum cohomology ring of OF. The Schubert classes σ_w, $w \in W^P$, form a basis. Let $\sigma^w := \sigma_w^\vee$ be the Poincare dual of σ_w for any $w \in W^P$. Let $\mathbb{Z}[q_1, q_2]$ be a polynomial ring where $\deg q_1 = 2$ and $\deg q_2 = 4n-8$. For a degree $d = (d_1, d_2)$ that corresponds to $d_1\sigma_{s_1} + d_2\sigma_{s_2} \in H_2(\mathrm{OF})$ (this is an integral sum of curve classes), we write $q^d = q_1^{d_1} q_2^{d2}$. The small quantum cohomology ring $\mathrm{QH}^*(\mathrm{OF})$ is a graded $\mathbb{Z}[q]$-module. The multiplication is given by

$$\sigma^v \star \sigma_w = \sum_{u, d \geqslant 0} c_{v^\vee, w}^{u,d} q^d \sigma_u$$

where $c_{v^\vee, w}^{u,d}$ is the Gromov-Witten invariant that enumerates the degree d rational curves.

4.1. **Fulton and Woodward's formula for minimal quantum degrees.** Given any element $\tau \in \mathrm{QH}^*(\mathrm{OF})$, we say that q^d **occurs** in τ if the coefficient of $q^d\sigma_w$ is not zero for some w. The following result provides an equivalent definition to degrees in terms of chains in the Bruhat graph.

Proposition 4.1. [FW04, Theorem 9.1] *Let $v, w \in W^P$, and let d be a degree. The following are equivalent:*

(1) There is a degree $c \leqslant d$ such that q^c occurs in $\sigma^v \star \sigma_w$.
(2) There is a chain of degree $c \leqslant d$ between v and w.

5. Maya diagrams

In this section, we will define Maya diagrams and give an associated characterization of the Bruhat order.

Definition 5.1. Let $w \in W^P$. The Maya diagram M^w corresponding to w is a $3 \times n$ grid with the southwest corner chosen to be the box $(1, 1)$ and we index with $(\text{rows}, \text{columns})$. We place an x in the $(j, w(i))$ position when $w(i) \leqslant n$ and $i \leqslant j \leqslant 2$. We place an $\bullet$ in the $(j, w(i))$ position when $w(i) \geqslant n$ and $i \leqslant j < 2$. Each box in the top row contains an x. We denote the row indexed by y as m_y^w.

Example 5.2. The minimal length representatives $w = (3|1)$ and $v = (\bar{2}|3)$ correspond to the Maya diagrams

$$M^w = \begin{array}{|c|c|c|c|c|} \hline x & x & x & x & x \\ \hline x & & x & & \\ \hline & & x & & \\ \hline \end{array} \text{ and } M^v = \begin{array}{|c|c|c|c|c|} \hline x & x & x & x & x \\ \hline & \bullet & x & & \\ \hline & \bullet & & & \\ \hline \end{array}.$$

Let M^w be the Maya diagram corresponding to $w \in W^P$ and let $1 \leqslant y \leqslant 2$. Let $\pi_y : W^P \to W^{P_y}$ denote the natural projection. Then $M^{\pi_y(w)}$ is a Maya diagram with two rows and n columns, with the top row having an x in each position and the bottom row is m_y^w.

Example 5.3. We have the following.

$$M^w = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline & \bullet & x & & \\ \hline & \bullet & & & \\ \hline \end{array}, M^{\pi_1(w)} = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline & \bullet & & & \\ \hline \end{array}, \text{ and } M^{\pi_2(w)} = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline & \bullet & x & & \\ \hline \end{array}.$$

5.1. **Bruhat order with Maya diagrams.** We begin the subsection with technical definitions.

Definition 5.4. Let $w, v \in W^P$. Let M^w be the Maya diagram that corresponds to $w \in W^P$.

(1) Define

$$f(M^w, y, z) := \begin{cases} x & z = w(i) \leqslant n \text{ for all } 1 \leqslant i \leqslant y; \\ \bullet & z = w(i) \geqslant \overline{n} \text{ for all } 1 \leqslant i \leqslant y; \\ 0 & \text{otherwise}; \end{cases}.$$

(2) Define $S_y(M^w, z) := \#\{i : f(M^w, y, i) \in \{x, \bullet\} \text{ for } 1 \leqslant i \leqslant z\}$.
(3) We say $M^w \leqslant M^v$ if $S_y(M^w, z) \geqslant S_y(M^v, z)$ for all y and z such that $1 \leqslant z \leqslant \overline{1}$ and $1 \leqslant y \leqslant 2$.
(4) Let $y \in \{1, 2\}$. We say $M^{\pi_y(w)} \leqslant M^{\pi_y(w)}$ if $S_y(M^w, z) \geqslant S_y(M^v, z)$ for all z such that $1 \leqslant z \leqslant \overline{1}$.

Example 5.5. Consider $w = (\bar{2}|3)$. Then we have

$$M^w = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline & \bullet & x & & \\ \hline & \bullet & & & \\ \hline \end{array}.$$

We have $f(M^w, 2, \bar{2}) = 2$, $f(M^w, 2, 3) = 1$, and $f(M^w, 1, 3) = 0$.

Example 5.6. Recall the Maya diagrams from Example 5.2.

$$M^w = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline & & x & & x \\ \hline & & & & x \\ \hline \end{array} \text{ and } M^v = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline & \bullet & x & & \\ \hline & \bullet & & & \\ \hline \end{array}.$$

Next we have a few examples of computations color coordinated to match with the x that are being counted in the Maya diagrams.

$$S_1(M^w, 5) = 1 \geqslant 0 = S_1(M^v, 5) \text{ and } S_2(M^w, \bar{3}) = 2 \geqslant 1 = S_2(M^v, \bar{3}).$$

In this example, we have $M^w \leqslant M^v$.

Next, we present a proposition that relates the Bruhat order in W^P with the partial order in Maya diagrams. This is another way of presenting the result in [Pro82, Theorem 5BC].

Proposition 5.7. [Pro82, Theorem 5BC] *Let $w, v \in W^P$. Then $w \leqslant v$ if and only if $M^w \leqslant M^v$.*

6. Rim hook rules

In this section, we will define four rim hook rules corresponding to degrees $(1,0)$, $(0,1)$, $(1,1)$, and $(1,2)$. These rules are the main tools for calculating minimum quantum degrees and proving their uniqueness. These rules correspond to the curve neighborhood calculations in [BM15].

6.1. $(1,0)$**-rim hook rule.** Let $v = (j|k) \in W^P$. If $j > k$ then the $(1,0)$-rim hook rule is not defined. If $j < k$ then the $(1,0)$-rim hook rule is defined by $(j|k) \xrightarrow{(1,0)} (k|j)$.

Example 6.1. We have the following example.

x	x	x	x	x
x			x	
x				

$\xrightarrow{(1,0)-\text{rim hook}}$

x	x	x	x	x
x			x	
			x	

.

6.2. $(0,1)$**-rim hook rule.** Let $v = (j|k) \in W^P \backslash \{(j|k) : k = \bar{1}\}$. Then the $(0,1)$-rim hook rule is defined by $(j|k) \xrightarrow{(0,1)} (j|k^*)$ where $k^* = \max\{v(2), v(3), \cdots, v(\bar{3})\}$. In terms of Maya diagrams, the $(0,1)$-rim hook of M^v is found using the following algorithm. First, note that if the leftmost box in the second row is $\bullet$, then the rim hook rule is not defined.

Algorithm 1 $(0,1)$-rim hook rule

If the leftmost box in the second row is $\bullet$ and the leftmost box in the first row is empty or if the two leftmost entries of the second row are both $\bullet$, then the $(0,1)$-rim hook rule is not defined.
(1) Place a $\bullet$ in the leftmost unoccupied box in the second row.
(2) Remove the other x or $\bullet$ in the second row that has an empty box below it in the first row.

Example 6.2. We have the following example.

x	x	x	x	x
x	x			
	x			

$\xrightarrow{(0,1)-\text{rim hook}}$

x	x	x	x	x
	x	$\bullet$		
	x			

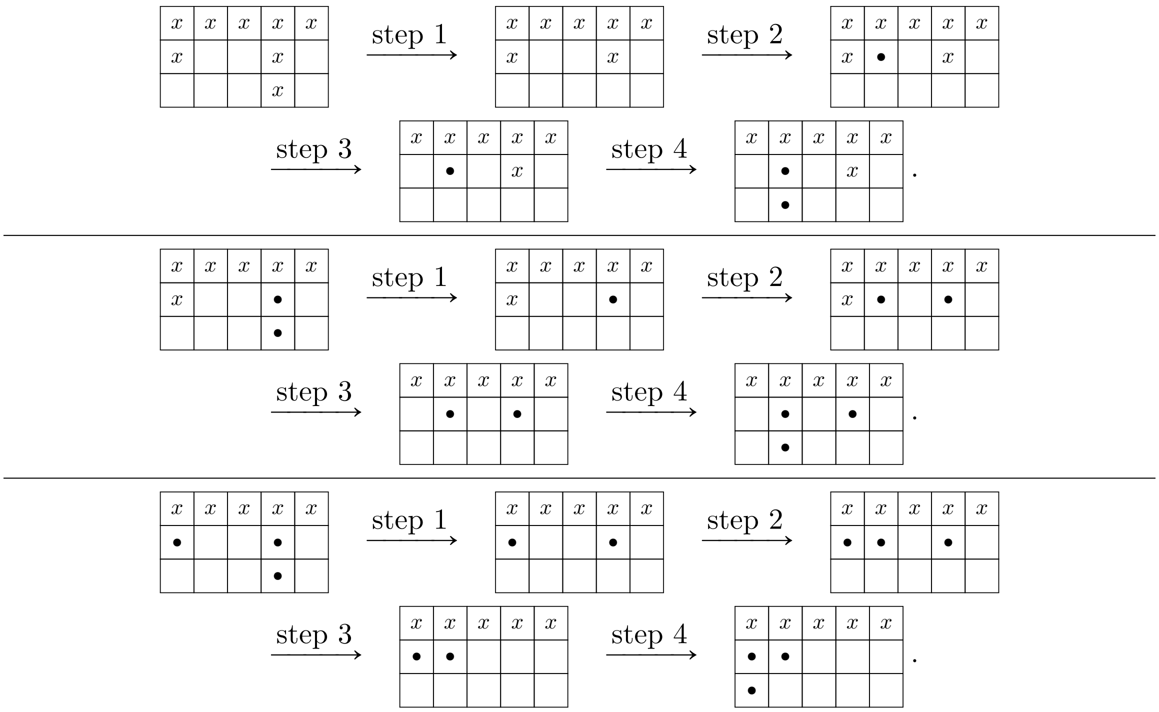
.

6.3. $(1,1)$**-rim hook rule.** Let $v = (j|k) \in W^p \backslash \{(\bar{1}|k), (\bar{2}|\bar{1}) : k \leqslant \bar{2}\}$, $j^* = \max\{v(1), v(2)\}$ and $k^* = \max\{v(3), v(4), \cdots, v(\bar{3})\}$. Define $j^{**} = \max\{j^*, k^*\}$ and $k^{**} = \min\{j^*, k^*\}$. Then $(j|k) \xrightarrow{(1,1)} (j^{**}, k^{**})$. In terms of Maya diagrams, the $(1,1)$-rim hook of M^v is found with the following algorithm. First, note that if the two leftmost entries in the second row are $\bullet$, then the rim hook rule is not defined.

Algorithm 2 $(1,1)$-rim hook rule

If the two leftmost entries of the second row are both $\bullet$ or if the leftmost entry in the first row is $\bullet$, then the $(1,1)$-rim hook rule is not defined.
(1) Delete the entry in the first row.
(2) Put a $\bullet$ in the leftmost empty box on the second row.
(3) If there is an x in the second row, then delete the leftmost x. If there is no x in the second row, delete the right most $\bullet$.
(4) Place a $\bullet$ in the first row under the leftmost $\bullet$ in the second row.

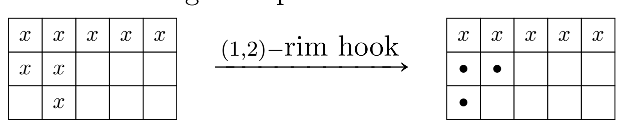
Example 6.3. Next, we will provide three examples of applying the $(1,1)$-rim hook rule to a Maya diagram.



6.4. $(1,2)$**-rim hook rule.** Let $v = (j|k) \in W^P$. Then the $(1,2)$-rim hook rule is defined by one of the following cases.

(1) $(j|k) \xrightarrow{(1,2)} (\overline{2}|\overline{1})$ when $j = 1$.
(2) $(j|k) \xrightarrow{(1,2)} (\overline{1}|\overline{2})$ when $j > 1$.

Example 6.4. We have the following example.



7. Additional Preliminary results with rim hook rules

The main results for this section connect the (a,b)-rim hook rules to degree (a,b)-chains in an explicit way. These results help control the combinatorics associated to minimum degree calculations. We begin with a technical definition.

Definition 7.1. Let $\{\lambda_q\}_{q=1}^{Q}, \{\beta_j\}_{j=1}^{J} \subset R^+$. We say that $\sum \lambda_q \geqslant \sum \beta_j$ if $\sum \lambda_q - \sum \beta_j$ is a non-negative linear combination of positive roots.

Lemma 7.2. *Let M^v be a Maya diagram that corresponds to $v \in W^P$. Let $(a,b) \in \{(1,0),(0,1),(1,1),(1,2)\}$. Apply the (a,b)-rim hook rule to M^v and call the resulting Maya diagram $M^{v_{(a,b)}}$ where $v_{(a,b)} \in W^P$. Then there exists a sequence of positive roots $\{\beta_j\}_{j=1}^{J} \subset R^+$ such that*

(1) $(a,b) \geqslant \sum \beta_j^\vee$.
(2) $v_{(a,b)} = v s_{\beta_1} s_{\beta_2} \ldots s_{\beta_J}$.

Proof. The proof is constructive, so we produce a sequence of positive roots for each possible case. If $(a, b) = (1, 0)$ then (1) and (2) are both clear.

Suppose $(a, b) = (0, 1)$. Since $v \in W^P$, one of the following must occur: If $v = (\bar{1}|k)$ with $k < \bar{2}$ then $v_{(a,b)} = vs_{t_2+t_3}$. If $v = (j|k)$ with $j, k < \bar{1}$ then $v_{(a,b)} = vs_{t_2+t_3}$. If $v = (k|\bar{1})$ then $v_{(a,b)}$ is not defined. If $v = (\bar{1}|\bar{2})$ then $v_{(a,b)}$ is not defined.

Suppose $(a, b) = (1, 1)$. Since $v \in W^P$ then one of the following must occur: If $v = (j|k)$ with $j, k < \bar{1}$ then $v_{(a,b)} = vs_{t_1+t_3}$. If $v = (j|\bar{1})$ with $j < \bar{2}$ then $v_{(a,b)} = s_{t_1-t_2}s_{t_2+t_3}$. If $v = (\bar{1}|k)$ then $v_{(a,b)}$ is not defined. If $v = (\bar{2}|\bar{1})$ then $v_{(a,b)}$ is not defined.

Suppose $(a, b) = (1, 2)$. Since $v \in W^P$, then one of the following must occur: If $v = (1|2)$ or $v = (2|1)$ then $v_{(a,b)} = vs_{t_1+t_2}$. If $v = (1|\bar{2})$ or $v = (2|\bar{1})$ then $v_{(a,b)} = vs_{t_1-t_2}s_{t_2}$. If $v = (\bar{2}|1)$ then $v_{(a,b)} = vs_{t_2}s_{t_1-t_2}$. If $v = (\bar{1}|2)$ then $v_{(a,b)} = vs_{t_2}$. If $v = (\bar{2}|\bar{1})$ then $v_{(a,b)} = vs_{t_1-t_2}$. If $v = (\bar{1}|\bar{2})$ then $v_{(a,b)} = v$. If $v = (1|k)$ with $\bar{2} > k > 2$ then $v_{(a,b)} = vs_{t_1+t_3}s_{t_2-t_3}$. If $v = (k|1)$ with $\bar{2} > k > 2$ then $v_{(a,b)} = vs_{t_2+t_3}s_{t_1-t_3}$. If $v = (\bar{1}|k)$ with $\bar{2} > k > 2$ then $v_{(a,b)} = vs_{t_2+t_3}$. If $v = (k|\bar{1})$ with $\bar{2} > k > 2$ then $v_{(a,b)} = vs_{t_1-t_2}s_{t_2+t_3}$. If $v = (2|k)$ with $\bar{2} > k > 2$ then $v_{(a,b)} = vs_{t_1+t_3}s_{t_2-t_3}$. If $v = (k|2)$ with $\bar{2} > k > 2$ then $v_{(a,b)} = vs_{t_2+t_3}s_{t_1-t_2}s_{t_2-t_3}$. If $v = (\bar{2}|k)$ with $\bar{2} > k > 2$, then $v_{(a,b)} = vs_{t_2+t_3}s_{t_1-t_2}$. If $v = (k|\bar{2})$ with $\bar{2} > k > 2$, then $v_{(a,b)} = vs_{t_1+t_3}$. If $v = (j|k)$ with $\bar{2} > j, k > 2$, then $v_{(a,b)} = vs_{t_1+t_3}s_{t_2+t_4}$. This completes the proof. ☐

As an immediate consequence of Lemma 7.2 we have the following proposition.

Proposition 7.3. *Let M^v be a Maya diagram that corresponds to $v \in W^P$. Apply the (a, b)-rim hook rule to M^v and call the resulting Maya diagram $M^{v_{(a,b)}}$ where $v \in W^P$. Then there is a chain $\mathcal{C}$ originating at v and terminating at $v_{(a,b)}$ such that $\deg_C(v, v_{(a,b)}) \leqslant (a, b)$.*

8. Minimum Quantum Degree Calculations

In this section, we prove the main result. The theorems are broken down by Bruhat compatibility in each row of Maya diagrams through natural projections π_y. Theorem 8.2 handles the case where both rows are compatible in the Bruhat order. Theorem 8.3 is the case where only the first row is not compatible in the Bruhat order. Theorem 8.4 is the case where the second row is not compatible in the Bruhat order. Theorem 8.5 is the case where neither row is compatible in the Bruhat order. It is important to note that in Theorem 8.5 the choice of whether to use the $(1, 1)$-rim hook or the $(1, 2)$-rim hook needs to be made, which distinguishes this work from the results in [Shi25]. We begin with a technical definition.

Definition 8.1. Define $\{l, m\} \lhd \{j, k\}$ to mean that $\min\{l, m\} \leqslant \min\{j, k\}$ and $\max\{l, m\} \leqslant \max\{j, k\}$.

Theorem 8.2. *Suppose that we have two Maya diagrams with $M^w \leqslant M^v$. Then $d = (0, 0)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.*

Proof. This is clear. ☐

Theorem 8.3. *Suppose that there are two Maya diagrams M^w and M^v where $M^{\pi_1(w)} \nleqslant M^{\pi_1(v)}$ but $M^{\pi_2(w)} \leqslant M^{\pi_2(v)}$. Then $d = (1, 0)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.*

Proof. Suppose $v = (j|k)$ and $w = (l|m)$. This means $j < l$ and $\{l, m\} \lhd \{j, k\}$. Assume $k < j$, then $k < j < l$. If $l < m$, then $\{l, m\} \ntriangleleft \{j, k\}$ as $m > j$ and $l > k$. If $m < k$, then $\{l, m\} \ntriangleleft \{j, k\}$ as $m < k < j < l$. Thus, $j < k$ and $l, m < k$ for $\{l, m\} \lhd \{j, k\}$. One element of $\{l, m\}$ must be less than j, but $j < l$, so $m < j$ and $m < j < l < k$ is the only possible ordering. Applying $d = (1, 0)$ to M^v, $(j|k) \to (j^*|k^*)$ where j^* is the maximum element of $\{j, k\}$. As shown above $j^* = k > l$ and $M^w < M^v$. Finally, note that the chain corresponding to the rim hook cannot be degree 0. This completes the proof. □

Theorem 8.4. *Suppose that two Maya diagrams M^w and M^v where $M^{\pi_2(w)} \nleqslant M^{\pi_2(v)}$ but $M^{\pi_1(w)} \leqslant M^{\pi_1(v)}$.*

(1) If $M^v \xrightarrow{(0,1)-\text{rim hook}} M^{v_{(0,1)}}$ with $M^w \leqslant M^{v_{(0,1)}}$ then $d = (0, 1)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.

(2) If $M^v \xrightarrow{(0,1)-\text{rim hook}} M^{v_{(0,1)}} \xrightarrow{(0,1)-\text{rim hook}} M^{v_{(0,2)}}$ with $M^w \nleqslant M^{v_{(0,1)}}$ and $M^w \leqslant M^{v_{(0,2)}}$ then $d = (0, 2)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.

Proof. This a direct consequence of Proposition 4.1 and Proposition 7.3 after noting the chains corresponding to each rim hook cannot be degree 0. This completes the proof. □

Theorem 8.5. *Suppose that two Maya diagrams M^w and M^v where $M^{\pi_2(w)} \nleqslant M^{\pi_2(v)}$ and $M^{\pi_1(w)} \nleqslant M^{\pi_1(v)}$.*

(1) If $M^v \xrightarrow{(1,1)-\text{rim hook}} M^{v_{(1,1)}}$ with $M^w \leqslant M^{v_{(1,1)}}$ then $d = (1, 1)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.

(2) If $M^v \xrightarrow{(1,1)-\text{rim hook}} M^{v_{(1,1)}}$ and $M^v \xrightarrow{(1,2)-\text{rim hook}} M^{v_{(1,2)}}$ with $M^w \nleqslant M^{v_{(1,1)}}$ and $M^w \leqslant M^{v_{(1,2)}}$ then $d = (1, 2)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.

(3) If $M^v \xrightarrow{(1,2)-\text{rim hook}} M^{v_{(1,2)}} \xrightarrow{(1,0)-\text{rim hook}} M^{v_{(2,2)}}$ with $M^w \nleqslant M^{v_{(1,2)}}$ and $M^w \leqslant M^{v_{(2,2)}}$ then $d = (2, 2)$ is the unique smallest d such that q^d occurs in the quantum product $\sigma^v \star \sigma_w$.

Proof. For this proof we must use care since Proposition 7.3 indicates that the application of the (a, b)-rim hook guarantees a chain of *at most* degree (a, b) originating at v and terminating at $v_{(a,b)}$.

We consider part (1). A chain from v to w in W^P of degree $(1, 0)$ does not exist since $M^{\pi_2(w)} \nleqslant M^{\pi_2(vs_1)}$. Likewise, a chain from v to w in W^P of degree $(0, 1)$ does not exist since $M^{\pi_2(w)} \nleqslant M^{\pi_2\left(vs_{t_2 \pm t_k}\right)}$ for an $3 \leqslant k \leqslant n$. By Proposition 7.3 the result for part (1) follows.

For part (2) note that the work for the proof of part (1) eliminates the possibility that no chain from v to w of degree $(1, 0)$ or $(0, 1)$ exists. By the proof of Lemma 7.2 we have the following: If $v = (j|k)$ with $j, k < \bar{1}$ then $v_{(a,b)} = vs_{t_1+t_3}$; if $v = (j|\bar{1})$ with $j < \bar{2}$ then $v_{(a,b)} = s_{t_1-t_2}s_{t_2+t_3}$; if $v = (\bar{1}|k)$ then $v_{(a,b)}$ is not defined; if $v = (\bar{2}|\bar{1})$ then $v_{(a,b)}$ is not defined. Notice that $t_1 + t_3$ is the maximal root of the set $\{\alpha \in R^+ \mid \alpha^\vee + \Delta_P^\vee = \alpha_1^\vee + \alpha_2^\vee + \Delta_P^\vee\}$, see Definition 7.1, and $t_1 + t_3 = (t_1 - t_2) + (t_2 + t_3)$. Therefore, there is no chain from v to w of degree $(1, 1)$. In the cases where the $(1, 1)$-rim hook does not exist, a chain from v to the longest permutation $(\bar{1}|\bar{2})$ is reached with a chain of degree $(0, 1)$, $(1, 0)$, or $(0, 0)$. By Proposition 7.3 the result for part (2) follows.

For part (3) first observe that $v_{(1,1)} \leqslant v_{(1,2)}$. If $M^w \nleqslant M^{v_{(1,2)}}$ then it must be the case that $v_{(1,2)} = (\bar{2}|\bar{1})$ and $w = (\bar{1}|\bar{2})$. So, only a $(1, 0)$-rim hook may be applied and the result follows. □

We conclude with a corollary that states which degrees appear as minimum degrees in quantum products.

Corollary 8.6. *For any $v, w \in W^P$, the unique minimum quantum degree that occurs in the quantum product $\sigma^v \star \sigma_w$ must be in set* $\{(0,0),(1,0),(0,1),(1,1),(1,2),(0,2),(2,2)\}$.

Proof. For any v note that $v_{(1,2)} = (\bar{1}|\bar{2})$ or $(\bar{2}|\bar{1})$. In the second case, note that $(v_{(1,2)})_{(1,0)} = (\bar{1}|\bar{2})$. That is, no degree larger than $(2,2)$ may appear as a minimum quantum degree. The degrees $(2,0)$ and $(2,1)$ will not appear as minimum quantum degrees because $v_{(1,0)}s_1 < v_{(1,0)}$ and $v_{(1,1)}s_1 < v_{(1,1)}$. The result follows. □

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