ON FROBENIUS-PERRON DIMENSION AND GALKIN'S LOWER BOUND CONJECTURE

RYAN M. SHIFLER AND STEPHANIE WARMAN

Abstract. Let F be a Fano variety and let $\mathrm{QH}^*(F)$ be the quantum cohomology ring of F. The quantum multiplication by a basis element σ of $H^*(F)$ induces an endomorphism $\hat{\sigma}$ of the finite-dimensional vector space $\mathrm{QH}^*(F)_{|q=1}$ specialized at $q=1$. We study this operator for partial flags, isotropic Grassmannians in types BCD, and for the full flag of type G_2. We discuss Frobenius-Perron dimension and state related conjectures. We give a numerical check that a conjecture by Galkin holds. It states that the largest real eigenvalue of the endomorphism $\hat{c}_1$ induced by the first Chern class is greater than or equal to $\dim F{+}1$ with equality if and only if $F=\mathbb{P}^{n-1}$.

1. Introduction

The purpose of this paper is to outline a series of conjectures related to Frobenius-Perron dimension on the quantum cohomology ring, specialized at $q=1$, of a homogeneous space and provide a numerical check of a conjecture by Galkin [Gal]. These conjectures are supported by experimental calculations. The particular homogeneous spaces that we will study are partial flags in Type A; isotropic Grassmannians in Types BCD; and the full flag for G_2. The study of Galkin's lower bound recently motivated a generalized notion of Frobenius-Perron dimension for certain free $\mathbb{Z}$-module of infinite rank [LSYZ]. We first state the main objects of interest.

Let F be a Fano manifold (we will consider the case $F=G/P$ throughout the paper). The quantum cohomology ring $(\mathrm{QH}^*(F),\star)$ is a graded algebra over $\mathbb{Z}[q]$, where q is the quantum parameter. The parameter $q=(q_1,q_2,\cdots,q_s)$. Consider the specialization $H^\bullet(F):=\mathrm{QH}^*(F)_{|q=1}$ at $q=1$ (here we mean $q_i=1$ for all i). The quantum multiplication by the class $\alpha\in\mathrm{QH}^*(F)$ induces an endomorphism $\hat{\alpha}$ of the finite-dimensional vector space $H^\bullet(F)$:

$$y\in H^\bullet(F)\mapsto\hat{\alpha}(y):=(\alpha\star y)|_{q=1}.$$

We first discuss Frobenius-Perron dimension.

1.1. **Frobenius-Perron Dimension.** In this manuscript we will explore the notion of Frobenius-Perron dimension for quantum cohomology rings of homogeneous spaces specialized at $q=1$. The notion of Frobenius-Perron dimension is motived by that of index of subfactor [Jon83]. It was first defined as functions with certain properties [FK93]. The theory of Frobenius-Perron dimensions for general fusion rings and categories was developed by Etingof, Nikshych, and Ostrick [ENO05].

We will follow [EGNO15]. Let A be a unital $\mathbb{Z}_+$-ring of finite rank. This means that A is a free finitely generated abelian group with basis $\{\beta_i\}_{i\in I}$ with a unital associative ring structure such that $\beta_i\beta_j=\sum_k c_{i,j}^r\beta_r$ with $c_{i,j}^r\in\mathbb{Z}_+$ and $\beta_0=1$ for some element $0\in I$.

Each β_i induces a linear operator $\hat{\beta}_i:A\to A;\ \gamma\mapsto\beta_i\cdot\gamma$. Let $\rho(\hat{\beta}_i)$ of $\hat{\beta}_i$ be defined by

$$\rho(\hat{\beta}_i)=\max\{|c|:c\text{ is an eigenvalue of }\hat{\beta}_i\}\in\mathbb{R}_{\geqslant 0}.$$

2010 *Mathematics Subject Classification.* Primary 14N35; Secondary 15B48, 14N15, 14M15.

It is know by Perron-Frobenius Theory of non-negative matrices that $\rho(\hat{\beta}_i)$ is an eigenvalue. We will now define Frobenius-Perron dimension of A.

Definition 1.1. *Let A be a $\mathbb{Z}_+$-ring of finite rank. The function $FPdim = FPdim_A$,*

$$FPdim : A \to \mathbb{C}; FPdim\left(\sum_i a_i\beta_i\right) := \sum_i a_i\rho(\hat{\beta}_i),$$

is called the **Frobenius-Perron dimension** *of A.*

The function FPdim : $A \to \mathbb{C}$ is a ring homomorphism in the case that A is transitive[1] and unital. We do not approach the question of whether or not $H^\bullet(G/P)$ is transitive in this manuscript.

Question 1.2. *We will explore the following questions.*

(1) *Is $FPdim : H^\bullet(G/P) \to \mathbb{C}$ a ring homomorphism?*
(2) *Can $FPdim(\sigma_w)$ be expressed as a transcendental function for all $\sigma_w \in H^\bullet(G/P)$?*

The questions are answered by [Rie01] for the Grassmannian $\mathrm{Gr}(m,n)$. The bases elements $\{\sigma_\lambda\}$ of $H^\bullet(\mathrm{Gr}(m,n))$ are Schubert classes and are indexed (by codimension) by partitions of the form $(n-m \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0)$. Indeed, the linear operators $\hat{\sigma}_\lambda$ on $H^\bullet(\mathrm{Gr}(m,n))$ can be simultaneously diagonalized (with respect to a common basis $\{\sigma_I\}$ therein).

Let $\mathrm{Fl}(a;n)$ be the partial flag, $\mathrm{IG}(n,2n)$ be the Lagrangian Grassmannian, and $\mathrm{OG}(n,2n+1)$ be the maximal orthogonal Grassmannian. It is known from [Rie01, Che17, CH] that

(1) $\mathrm{FPdim}(\sigma_\lambda) = \dfrac{\Pi_{(i,j)\in\lambda}\sin((m-i+j)\frac{\pi}{n})}{\Pi_{(i,j)\in\lambda}\sin((\lambda_i+\lambda_j^t-i-j+1)\frac{\pi}{n})}$ for $\mathrm{Gr}(m,n)$;
(2) $\mathrm{FPdim}(\sigma_{(1)}) = 2^{\frac{-1}{n+1}}\sin\left(\dfrac{\pi}{2(n+1)}\right)^{-1}$ for $\mathrm{IG}(n,2n)$;
(3) $\mathrm{FPdim}(\sigma_{(1)}) = \dfrac{2^{\frac{1}{n}}}{2}\sin\left(\dfrac{\pi}{2n}\right)^{-1}$ for $\mathrm{OG}(n,2n+1)$.

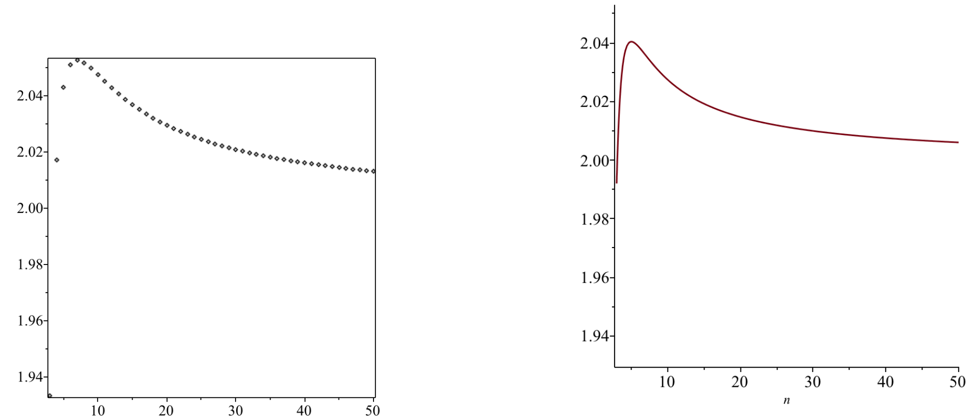
Observe that FPdim can be considered as a single variable function of n in the cases listed above (e.g. m is fixed in the $\mathrm{Gr}(m,n)$ case). Moreover, each function of n extends to a transcendental function on $\mathbb{R}^+$.

Very little is known in other cases. Quantum cohomology rings are usually considered on a case-by-case basis since they are not functorial. The novelty of this paper is that we plot FPdim as a function of n (with all other parameters fixed) to see that it's reasonable to conjecture that the function extends to a transcendental function on $\mathbb{R}^+$. This is motivated by the work in [ESS+, LSYZ]. We are ready to state our main conjecture pertaining to Frobenius-Perron dimension.

Conjecture 1. *Let $X \in \{\mathrm{Fl}(a;n), \mathrm{OG}(m,2n+1), \mathrm{IG}(m,2n), \mathrm{OG}(m,2n+2)\}$ and consider $FPdim(\hat{\sigma}_\lambda)$ where λ is an element of the appropriate index set.*

(1) *The function $FPdim : H^\bullet(X) \to \mathbb{C}$ a ring homomorphism;*
(2) *When considered as a function of n, $FPdim(\sigma_\lambda)$ extends to a transcendental function on $\mathbb{R}^+$;*
(3) *The number $FPdim(\sigma_\lambda)$ can be expressed as a transcendental function;*

[1] The ring A is transitive when, for any j, r, there exits an i such that $c_{i,j}^r > 0$.



(A) $\mathrm{FPdim}(\sigma_{(1)})$ of $\mathrm{IG}(2, 2n)$ for $3 \leqslant n \leqslant 50$.

(B) $f_k(x) = \dfrac{2^{-\frac{1}{2x+1-2}} \sin\left(\frac{k\pi}{2(2x-2)}\right)}{\sin\left(\frac{\pi}{2(2x+1-2)}\right)}$

FIGURE 1. The shape of the function $f_k(x)$ is reasonably correct, however, it does not match the calculate eigenvalues.

(4) *In particular, FPdim*(σ_λ) *is a product and quotient of the transcendental functions*

$$2^{\left(\frac{1}{g_1}\right)}, \sin\left(\frac{m\pi}{g_2}\right), \sin\left(\frac{\pi}{g_3}\right)$$

where g_1, g_2, g_3 *are functions of* λ, a, m, *and* n.

We expand on Conjecture 1 in Section 3. The Appendix (See 5) contains examples of FPdim plotted as a function of n for partial flags and isotropic Grassmannians of Types BCD.

Example 1.3. Figures 1 gives a first example and the graph of a function using the transcendental functions listed in Conjecture 1.

Next we will discussion Galkin's lower bound conjecture in the context of Frobenius-Perron dimension.

1.2. **Galkin Lower Bound Conjecture.** We next recall the precise statements of the conjecture by Galkin, following [Gal] and [GGI16, §3]. Let $K := K_F$ be the canonical bundle of F and let $c_1(F) := c_1(-K) \in H^2(F)$ be the anticanonical class. The quantum multiplication by the first Chern class $c_1(F)$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $H^\bullet(F)$. Galkin's lower bound conjecture [Gal] states the following:

$$\mathrm{FPdim}(c_1) \geqslant \dim_{\mathbb{C}} F + 1 \text{ with equality if and only if } F = \mathbb{P}^N.$$

The conjecture was verified for the Grassmannian [ESS+], Lagrangian and Orthogonal Grassmannian [CH], del Pezzo surfaces (implicitly calculated)[HKLY], projective complete intersections [Ke], and the Cayley Grassmannian [BM].

In Section 4 we state the cases where Galkin's lower bound conjecture has been checked for partial flags, isotropic Grassmannians in types BCD. We also prove the conjecture for the full flag of Type G_2.

1.3. **Computational tools.** For computations we use a Maple program written by Buch (see [Buc]) and a program written in Python (see [War]). Buch's program calculates the quantum cohomology ring for Grassmannians in Types ABCD and with little effort one can also compute Frobenius-Perron dimension. The program written in Python is for doing computations for partial flag varieties.

Acknowledgements

The first named author would like to thank Leonardo Mihalcea and Changzheng Li for useful discussions and collaborations on related projects. We would like thank Anders Buch for writing the Maple progam Quantum Calculator and making it publicly available. We would also like to thank Joseph Anderson for useful discussions in software implementation.

2. Preliminaries

Let $X = G/P$. The small quantum cohomology is defined as follows. Let (σ_w) be a basis for the cohomology ring $H^*(X)$ and let $(\sigma_w^\vee)_i$ be the dual basis for the Poincaré pairing. The multiplication is given by

$$\sigma_w \star \sigma_v = \sum_{d \geqslant 0, k} c_{i,j}^{k,d} q^d \sigma_u$$

where $c_{w,v}^{u,d}$ are the 3-point, genus 0, Gromov-Witten invariants corresponding to rational curves of degree d intersection the classes σ_w, σ_v, and $\sigma_u^\vee$. We will state the quantum Chevalley formula for partial flag varieties and the quantum Pieri formula of isotropic Grassmannians of Types BCD and for the full flag in Type G_2.

2.1. **Partial flag varieties.** We will follows the exposition of [Buc05, Buc15]. Let n be a positive integer. Given a strictly increasing sequence of integers $a = (a_1 < a_2 < \cdots < a_k)$ with $a_1 > 0$ and $a_k < n$, we let $\mathrm{Fl}(a; n)$ be the variety of partial flags $V_{a_1} \subset V_{a_2} \subset \cdots \subset V_{a_k} \subset \mathbb{C}^n$ where $\dim V_{a_j} = a_j$. For convenience we set $a_0 = 0$ and $a_{k+1} = n$. The dimension of $\mathrm{Fl}(a; n)$ is equal to $\sum_{i=1}^k a_i(a_{i+1} - a_i)$.

2.1.1. *Quantum cohomology of partial flag varieties.* Let S_n be the group of permutations of n elements. The Schubert varieties in $\mathrm{Fl}(a; n)$ are indexed by the set S_n/W_a, where $W_a \subset S_n$ is the subgroup generated by the simple transpositions $s_i = (i, i+1)$ for $i \notin \{a_1, \cdots, a_k\}$. Let $S_n(a) \subset S_n$ denote the set of permutations whose descent positions are contained in the set $\{a_1, a_2, \cdots, a_k\}$. These permutations are the shortest representatives for the elements of S_n/W_a.

The Schubert classes $\sigma_w^{(a)}$ form a basis for the cohomology ring $H^*(\mathrm{Fl}(a; n))$. Let $w_0 = n \cdots 21$ is the longest permutation in S_n, and w_a is the longest permutation in the subgroup $W_a \subset S_n$. That is, $w_a(j) = a_i + a_{i+1} + 1 - j$ for $a_i < j \leqslant a_{i+1}$.

Let $q_1, \cdots, q_k$ be independent variables and write $\mathbb{Z}[q] = \mathbb{Z}[q_1, \cdots, q_k]$. The (small) quantum cohomology ring $QH^*(\mathrm{Fl}(a; n))$ is a $\mathbb{Z}[q]$-algebra, which is a $\mathbb{Z}[q]$-module is free with a basis of quantum Schubert classes $\sigma_w^{(a)}$.

Multiplication is defined by the formula

$$\sigma_u^{(a)} \cdot \sigma_v^{(a)} = \sum_{w,d} c_{u,v}^{w^\vee, d} q^d \sigma_w^{(a)}$$

where the sum is over all $w \in S_n(a)$ and the multidegrees d, and $q^d = q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k}$. The ring $QH^*(\mathrm{IF}(a;n))$ has a natural grading where the degree of $\sigma_w^{(a)}$ is the length $\ell(w)$, while each variable q_i has degree $a_{i+1} - a_{i-1}$.

2.1.2. *Quantum Chevalley Formula.* We will now introduce additional notation to state the Chevalley formula for $\mathrm{Fl}(a;n)$. Let (i,j) denote the transposition interchanging i and j.

Definition 2.1. *Consider the simple reflection $\alpha = s_r$. For permutations u and w we write $u \xrightarrow{\alpha} w$ if there exists integers b and c such that*

(1) $b \leqslant r < c$;
(2) $w = u(b,c)$;
(3) $\ell(u(b,c)) = \ell(u) + 1$.

We call a sequence $d = (d_1, \cdots, d_k)$ of non-negative integers a *Pieri sequence with maximum position* j, if $(d_1, \cdots, d_j)$ is weakly increasing, $(d_j, \cdots, d_k)$ is weakly decreasing, and if we set $d_0 = d_{k+1} = 0$ then $|d_i - d_{i+1}| \leqslant 1$ for $0 \leqslant i \leqslant k$.

Let d be a non-zero Pieri sequence with maximum position j of value 1 where the first 1 appears in position r_1 and the last 1 appears in position r_2. Let $w_0(a)$ be the longest element in S_n. The define $s_d \in S_n$ to be $s_d := w_0 w_0^{(a)}(a_{r_1}, a_{r_2} + 1)$.

Theorem 2.2 (Quantum Chevalley formula)**.** *Let $\alpha = s_{a_j}$ and $u \in S_n(a)$ be permutations. Then*

$$\sigma_\alpha^{(a)} \cdot \sigma_u^{(a)} = \sum_w \sigma_w^{(a)} + \sum_w q^d \sigma_w^{(a)}$$

where the first sum is over w such that $u \xrightarrow{\alpha} w$ and the second sum is over w such that $u s_d W_a = w W_a$ and $\ell(u W_a) + 1 = \ell(u s_d W_a) + \sum_{i=1}^{k} d_i (a_{i+1} - a_{i-1})$.

2.2. **Type B Grassmannian.**

2.3. **Notation.** We follow the exposition of [BKT09] to state the Pieri formulas in Types BCD. Let the set of k-strict partitions be denoted by

$$\mathcal{P}(k,n) := \{(n + k \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0) : \lambda_j \geqslant k \implies \lambda_{j+1} < \lambda_j\}$$

Let λ be a k-strict partition. Let $|\lambda| = \sum \lambda_i$ and $\ell(\lambda)$ be the number of nonzero parts. We will say that the box in row r and column c of λ is k-related to the box in row r' and column c' if $|c - k - 1| + r = |c' - k - 1| + r'$.

Definition 2.3. *For any two k-strict parition λ and μ, we have the relation $\lambda \to \mu$ if μ can be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that*

(1) *if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \backslash \lambda$;*
(2) *if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column much each be k-related to exactly one box of $\mu \backslash \lambda$, and these boxes of $\mu \backslash \lambda$ much all lie in the same row.*

If $\lambda \to \mu$, we let $\mathbb{A}$ be the set of boxes $\mu \backslash \lambda$ in columns $k+1$ through $k+n$ which are not mentioned in (1) or (2). Then define $N(\lambda, \mu)$ to be the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$. Here two boxes are connected if they share at least a vertex.

Let $\mathcal{P}'(k,n+1)$ be the set of $\nu \in \mathcal{P}(k,n+1)$ such that $\ell(v) = n+1-k$, $2k \leqslant \nu_1 \leqslant n+k$ and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 1$. For any $\nu \in \mathcal{P}'(k,n+1)$, we let $\tilde{\nu} \in \mathcal{P}(k,n)$ be the partition obtained by removing the first row of ν as well as $n+k-\nu_1$ boxes from the first column. That is,

$$\tilde{\nu} = (\nu_2, \nu_3, \cdots, \nu_r)$$

where $r = \nu_1 - 2k + 1$. Finally, let $\lambda \in \mathcal{P}(k,n)$ and define by $\lambda^* = (\lambda_2, \lambda_3, \cdots)$.

2.4. **Quantum cohomology of** $\mathrm{OG}(n-k, 2n+1)$**.** Consider the vector space $V \cong \mathbb{C}^{2n+1}$ and a nondegenerate form symmetric bilinear form on V. For each $m = n-k < n$, the odd orthogonal Grassmannian $\mathrm{OG}(m, 2n+1)$ parameterizes m-dimensional isotropic subspaces in V. The algebraic variety has dimension $\dim \mathrm{OG}(m, 2n+1) = 2m(n-m) + m(m+1)/2$. Moreover, the Schubert classes σ_λ are indexed by the set of k-strick partitions in $\mathcal{P}(k,n)$ and form a $\mathbb{Z}$-basis for the cohomology ring $H^*(\mathrm{OG}(m, 2n+1))$. The quantum cohomology ring $QH^*(\mathrm{OG}(m; 2n+1))$ has a natural grading where the degree of σ_λ is the length $|\lambda|$, the variable q has degree $2n-m$.

Theorem 2.4 (Quantum Pieri Formula)**.** *For any k-strict partition $\lambda \in \mathcal{P}(k,n)$ and integer $p \in [1, n+k]$, we have*

$$\sigma_p \star \sigma_\lambda = \sum_{\lambda \to \mu} 2^{N'(\lambda,\mu)} \sigma_\mu + \sum_{\lambda \to \nu} 2^{N'(\lambda,\nu)} \sigma_{\tilde{\nu}} q + \sum_{\lambda^* \to \rho} 2^{N'(\lambda^*,\rho)} \sigma_{\rho^*} q^2$$

in the quantum cohomology ring $QH^*(\mathrm{OG}(n-k, 2n+1))$. The first sum is classical; the second sum is over $\nu \in \mathcal{P}'(k, n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$; the third sum is empty unless $\lambda_1 = n+k$, and over $\rho \in \mathcal{P}(k,n)$ such that $\rho_1 = n+k$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - k + p$; and $N'(\lambda,\mu) = N(\lambda,\mu)$ if $p \leqslant k$ and $N'(\lambda,\mu) = N(\lambda,\mu) - 1$ if $p > k$.

2.5. **Type C Grassmannian.** Fix a vector space $V \cong \mathbb{C}^{2n}$ with a non-degenerate skew-symmetric bilinear form $(,)$, and fix a non-negative integer $m \leqslant n$. Let $\mathrm{IG}(m, 2n)$ parametrize m-dimensional isotropic subspaces of V. The dimension of this algebraic varieties is $\dim \mathrm{IG}(m, 2n+1) = 2m(n-m) + m(m+1)/2$ (The same as $\mathrm{OG}(m, 2n+1)$. Let $k = n-m$. Moreover, the Schubert classes σ_λ are indexed by the set of k-strick partitions in $\mathcal{P}(k,n)$ and form a $\mathbb{Z}$-basis for the cohomology ring $H^*(\mathrm{IG}(m, 2n))$. The quantum cohomology ring $QH^*(\mathrm{IG}(m; 2n))$ has a natural grading where the degree of σ_λ is the length $|\lambda|$, the variable q has degree $2n+1-m$ when $m < n$ and degree $2n$ when $m = n$.

Theorem 2.5 (Quantum Pieri Formula)**.** *For any k-strict partition $\lambda \mathcal{P}(k,n)$ and integer $p \in [1, n+k]$, we have*

$$\sigma_p \star \sigma_\lambda = \sum_{\lambda \to \mu} 2^{N(\lambda,\mu)} \sigma_\mu + \sum_{\lambda \to \nu} 2^{N(\lambda,\nu)-1} \sigma_{\nu^*} q$$

in the quantum cohomology ring of $\mathrm{IG}(n-k, 2n)$*. The first sum is over partitions $\mu \in \mathcal{P}(k,n)$ such that $|\mu| = |\lambda| + p$, and the second sum is over partitions $\nu \in \mathcal{P}(k, n+1)$ with $|\nu| = |\lambda| + p$ and $\nu_1 = n+k+1$.*

2.6. **Type D Grassmannian.** We consider the even orthogonal Grassmannian $\mathrm{OG}(m, 2n+2)$, which parametrizes the m-dimensional isotropic subspaces in a vector space $V \cong \mathbb{C}^{2n+1}$ with a nondegenerate symmetric bilinear form. The dimension of this algebraic varieties is $\dim \mathrm{OG}(m, 2n+2) = 2m(n+1-m) + m(m-1)/2$.

Let $k = n+1-m$. To any k-strict partition $\lambda \in \mathcal{P}(k,n)$ we associate a number in $\{0, 1, 2\}$ called the type of λ, denote $\mathrm{type}(\lambda)$. If λ has no part equal to k, then we set $\mathrm{type}(\lambda) = 0$;

otherwise we have $\mathrm{type}(\lambda) = 1$ or $\mathrm{type}(\lambda) = 2$. Define $\tilde{\mathcal{P}}(k,n)$ to be the k-strict partitions $\lambda \in \mathcal{P}(k,n)$ of all three types.

Moreover, the Schubert classes σ_λ are indexed by the set of k-strick partitions in $\tilde{\mathcal{P}}(k,n)$ and form a $\mathbb{Z}$-basis for the cohomology ring $H^*(\mathrm{OG}(m,2n+2))$. The quantum cohomology ring $QH^*(\mathrm{OG}(m;2n+2))$ has a natural grading where the degree of σ_λ is the length $|\lambda|$. For notation when type$(\lambda \neq 0$, we will say that $\sigma(\lambda)$ corresponds to the case $\mathrm{type}(\lambda) = 1$ and $\sigma'(\lambda)$ corresponds to the case $\mathrm{type}(\lambda) = 2$. the variable q has degree $2n+1-m$.

Given a k-strict partition λ, we say that the box in row r and column c of λ is k'-related to the box in row r' and column c' if $|c-(2k+1)/2| + r = |c'-(2k+1)/2| + r'$. Using this convention, the relation $\lambda \to \mu$ is defined as in Definition 2.3, with the added condition that $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$.

Let $g(\lambda,\mu)$ be the number of columns of μ among the first k which do not have more boxes than the corresponding columns of λ, and

$$h(\lambda,\mu) = g(\lambda,\mu) + \max\{\mathrm{type}(\lambda), \mathrm{type}(\mu)\}.$$

If $p \neq k$, then set $\delta_{\lambda\mu} = 1$. If $p = k$ and $N'(\lambda,\mu) > 0$, then set

$$\delta_{\lambda\mu} = \lambda'_{\lambda\mu} = 1/2,$$

while $N'(\lambda,\mu) = 0$ define $\delta_{\lambda\mu} = 1$ if $h(\lambda,\mu)$ is odd, otherwise $\delta_{\lambda\mu} = 0$. Likewise, define $\delta'_{\lambda\mu} = 1$ if $h(\lambda,\mu)$ is even, otherwise $\delta'_{\lambda\mu} = 0$.

Let $\tilde{\mathcal{P}}'(k,n+1)$ be the set of $\nu \in \tilde{\mathcal{P}}(k,n+1)$ such that $\ell(\nu) = n+2-k$, $2k-1 \leqslant \nu_1 \leqslant n+k$, and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 2$. For any $\nu \in \tilde{\mathcal{P}}'(k,n+1)$, we let $\tilde{\nu} = (\nu_2, \nu_3, \cdots, \nu_r)$ where $r = \nu_1 - 2k + 1$. Moreover, we have $\mathrm{type}(\tilde{v}) = \mathrm{type}(v)$, if $\mathrm{type}(v) = 0$, otherwise $\mathrm{type}(\tilde{v}) = 3 - \mathrm{type}(v)$. Finally, for any $\lambda \in \tilde{\mathcal{P}}(k,n)$, we define λ^* with $\mathrm{type}(\lambda^*) = \mathrm{type}(\lambda)$ by $\lambda^* = (\lambda_2, \lambda_3, \cdots)$.

Theorem 2.6 (Quantum Pieri Formula)**.** *For any k-strict partition $\lambda \in \tilde{\mathcal{P}}(k,n)$ and integer $p \in [1, n+k]$, we have*

$$\sigma_p \star \sigma_\lambda = \sum_{\lambda\to\mu} \delta_{\lambda\mu} 2^{N'(\lambda,\mu)} \sigma_\mu + \sum_{\lambda\to\nu} \delta_{\lambda\nu} 2^{N'(\lambda,\nu)} \sigma_{\tilde{\nu}} + \sum_{\lambda^*\to\rho} \delta_{\lambda^*\rho} 2^{N'(\lambda^*,\rho)} \sigma_{\rho^*} q^2$$

in the quantum cohomology ring of $QH^*(\mathrm{OG}(n+1-k, 2n+2))$. Here the first sum is overall $\mu \in \tilde{\mathcal{P}}(k,n)$ with $\lambda \to \mu$ and $|\mu| = |\lambda| + p$; the second sum is over $\nu \in \tilde{\mathcal{P}}'(k,n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$; and the third sum is empty unless $\lambda_1 = n+k$, and over $\rho \in \tilde{\mathcal{P}}(k,n)$ such that $\rho_1 = n+k$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - k + p$. Also,$N'(\lambda,\mu) = N(\lambda,\mu)$ if $p \leqslant k$ and $N'(\lambda,\mu) = N(\lambda,\mu) - 1$ if $p > k$. Furthermore, the product $\sigma'_k \star \sigma_\lambda$ is obtained by replacing δ with δ' throughout.

2.7. **Type G_2 flag.** Let Fl_G denote the flag of type G_2. The dimension of $\mathrm{Fl}_G = 6$. The Weyl group of G_2 indexed by elements of the dihedral group with 12 elements. That is, the Schubert classes σ_w are indexed by reduced words $w \in \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^6 = 1 \rangle$ and form a $\mathbb{Z}$-basis for the cohomology ring $H^*(\mathrm{Fl}_G)$. The quantum cohomology ring $QH^*(\mathrm{Fl}_G)$ has two quantum parameters q_1 and q_2 and the degree of both is equal to 2. Next we state the quantum Chevalley formula.

w	$\sigma_w \star \sigma_{s_1}$	$\sigma_w \star \sigma_{s_2}$
s_1	$\sigma_{s_2s_1} + q_1$	$\sigma_{s_1s_2} + \sigma_{s_2s_1}$
s_2	$\sigma_{s_1s_2} + \sigma_{s_2s_1}$	$3\sigma_{s_1s_2} + q_2$
s_1s_2	$\sigma_{s_1s_2s_1} + \sigma_{s_2s_1s_2}$	$2\sigma_{s_2s_1s_2} + q_2\sigma_{s_1}$
s_2s_1	$2\sigma_{s_1s_2s_1} + q_1\sigma_{s_2}$	$3\sigma_{s_1s_2s_1} + \sigma_{s_2s_1s_2}$
$s_1s_2s_1$	$\sigma_{s_2s_1s_2s_1} + q_1\sigma_{s_1s_2} + q_1q_2$	$\sigma_{s_1s_2s_1s_2} + 2\sigma_{s_2s_1s_2s_1} + q_1q_2$
$s_2s_1s_2$	$\sigma_{s_2s_1s_2s_1} + 2\sigma_{s_1s_2s_1s_2}$	$3\sigma_{s_1s_2s_1s_2} + q_2\sigma_{s_2s_1}$
$s_1s_2s_1s_2$	$\sigma_{s_1s_2s_1s_2s_1} + \sigma_{s_2s_1s_2s_1s_2}$	$\sigma_{s_2s_1s_2s_1s_2} + q_2\sigma_{s_1s_2s_1}$
$s_2s_1s_2s_1$	$\sigma_{s_1s_2s_1s_2s_1} + q_1\sigma_{s_2s_1s_2} + q_1q_2\sigma_{s_2}$	$\sigma_{s_2s_1s_2s_1s_2} + 3\sigma_{s_1s_2s_1s_2s_1} + q_1q_2\sigma_{s_2}$
$s_1s_2s_1s_2s_1$	$q_1\sigma_{s_1s_2s_1s_2} + q_1q_2\sigma_{s_1s_2}$	$\sigma_{w_0} + q_1q_2\sigma_{s_1s_2}$
$s_2s_1s_2s_1s_2$	$\sigma_{w_0} + q_1q_2^2$	$q_2\sigma_{s_2s_1s_2s_1} + 2q_1q_2^2$
$w_0 = s_1s_2s_1s_2s_1s_2$	$q_1\sigma_{s_2s_1s_2s_1s_2} + q_1q_2s_{s_2s_1s_2} + q_1q_2^2\sigma_{s_1}$	$q_2\sigma_{s_1s_2s_1s_2s_1} + q_1q_2\sigma_{s_2s_1s_2} + 2q_1q_2^2\sigma_{s_1}$

3. On Frobenius-Perron Dimension

3.1. **Is FPdim : $H^\bullet(G/P) \to \mathbb{C}$ a ring homomorphism?** We can make some progress toward answering Question 1.2. Let $X \in \{\mathrm{Fl}(a,n), \mathrm{OG}(m, 2n+1), \mathrm{IG}(m, 2n), \mathrm{OG}(m, 2n+2)\}$. We can better understand $\mathrm{FPdim}(\sigma_w + \sigma_v)$ and $\mathrm{FPdim}(\sigma_w\sigma_v)$ by considering $\hat{\sigma}_w + \hat{\sigma}_v$ and $\hat{\sigma}_w\hat{\sigma}_v$. We begin with a well-known elementary result.

Lemma 3.1. *Any two diagonalizable endomorphisms of the finite vector space V are simultaneously diagonalizable if and only they commute.*

Consider $\sigma_w, \sigma_v \in H^\bullet(G/P)$, the linear operators they induce, and assume they are diagonalizable. There exists a P such that $\hat{\sigma}_w = PD_1P^{-1}$ and $\hat{\sigma}_v = PD_2P^{-1}$ where D_1 and D_2 are diagonalizable. Then we have that

$$\begin{aligned} \hat{\sigma}_w\hat{\sigma}_v &= PD_1D_2P^{-1} \\ \hat{\sigma}_w + \hat{\sigma}_v &= P(D_1 + D_2)P^{-1}. \end{aligned}$$

It then suffices to show that the numbers $\mathrm{FPdim}(\sigma_w)$ and $\mathrm{FPdim}(\sigma_w)$ are in the same entry of D_1 and D_2, respectively. Thus, one needs to better understand how eigenvalues correspond to eigenvectors. We can reframe the point of view of the question "Is FPdim : $H^\bullet(G/P) \to \mathbb{C}$ a ring homomorphism?" We do this with the next conjecture.

Conjecture 2. *Let $\{\sigma_w\} \subset H^\bullet(X)$ be the Schubert basis and let $\lambda_w := FPdim(\sigma_w)$.*

(1) *Each eigenvalue of $\hat{\sigma}_w$ has multiplicity one for each w (i.e. $\hat{\sigma}_w$ is diagonalizable);*
(2) *Let E_{λ_w} be the eigenspace corresponding to $\hat{\sigma}_w$ and the eigenvalue λ_w. Then $E_{\lambda_w} = E_{\lambda_v}$ for any w, v.*

This subsection considered the ring homomorphism point of view to understand Frobenius-Perron dimension. In the next subsection we will consider extending FPdim to a real valued function to better understand its behavior.

3.2. **FPdim as a real function.** In this subsection we will expand on the second and third part of Conjecture 1. Consider the function

$$F_{m,\sigma}(n) := \mathrm{FPdim}(\sigma)$$

where $X \in \{\mathrm{OG}(m, 2n+1), \mathrm{IG}(m, 2n), \mathrm{OG}(m, 2n+2)\}$ and $\sigma \in H^\bullet(X)$ is a Schubert class. For the case of the partial flags $\mathrm{Fl}(a; n)$ consider

$$F_{a,\sigma}(n) := \mathrm{FPdim}(\sigma).$$

For notational ease we will use F instead of $F_{m,\sigma}$ or $F_{a,\sigma}$. Recall the following Conjecture from the introduction.

Conjecture 3. *For any* $X \in \{\mathrm{Fl}(a,n), \mathrm{OG}(m,2n+1), \mathrm{IG}(m,2n), \mathrm{OG}(m,2n+2)\}$ *the function* $F(n)$ *extends to a second transcendental function* $F(x) : \mathbb{R}^+ \to \mathbb{R}$.

Remark 3.2. A word of warning! In the Grassmannian case $\mathrm{Gr}(m,n)(= \mathrm{Fl}(m;n))$ it was proven in [LSYZ] that $F(n)$ is strictly increasing for $x > M$ for some M and concave down for $x > N$ for some N. It's tempting to make the same conjecture for partial flags when one considers the included graphics in Figure 3. However, we do not have enough evidence to make a similar claim for partial flags. Indeed, consider $\mathrm{Fl}(1,5;n)$ and $\sigma_{s_1} \in H^\bullet(\mathrm{Fl}(1,5;n))$. Here $F(16) = 1.56234... > F(20) = 1.56086....$ Thus we omit $\mathrm{Fl}(a,n)$ from our next Conjecture.

Our numerical results do support the following conjecture.

Conjecture 4. *For any* $X \in \{\mathrm{OG}(m,2n+1), \mathrm{IG}(m,2n), \mathrm{OG}(m,2n+2)\}$

(1) *The limit* $\lim_{x\to\infty} F(x)$ *exists;*
(2) *The function* $F(x)$ *is concave up for* $x > M$ *for some* M.

The study of Frobenius-Perron dimension for $H^\bullet(F)$ where F is a Fano variety is related to previous work understanding Galkin's lower bound conjecture.

4. On Galkin's lower bound

In this section we will state all the case where we have verified that Galkin's Lower Bound Conjecture holds. We begin by stating the first Chern classes of each homogeneous space that we are considering.

Lemma 4.1. *The first Chern classes are*

$$\begin{aligned}
c_1(\mathrm{IF}(a;n)) &= \sum_{i=1}^{k} \deg q_i \sigma^{(a)}_{s_{a_i}} = \sum_{i=1}^{k} (a_{i+1} - a_{i-1}) \sigma^{(a)}_{s_{a_i}} \\
c_1(\mathrm{OG}(m,2n+1)) &= (2n-m)\sigma_{(1)} \\
c_1(\mathrm{IG}(m,2n)) &= (2n+1-m)\sigma_{(1)} \\
c_1(\mathrm{OG}(m,2n+2)) &= (2n+1-m)\sigma_{(1)} \\
c_1(\mathrm{Fl}_G) &= 2\sigma_{s_1} + 2\sigma_{s_2}.
\end{aligned}$$

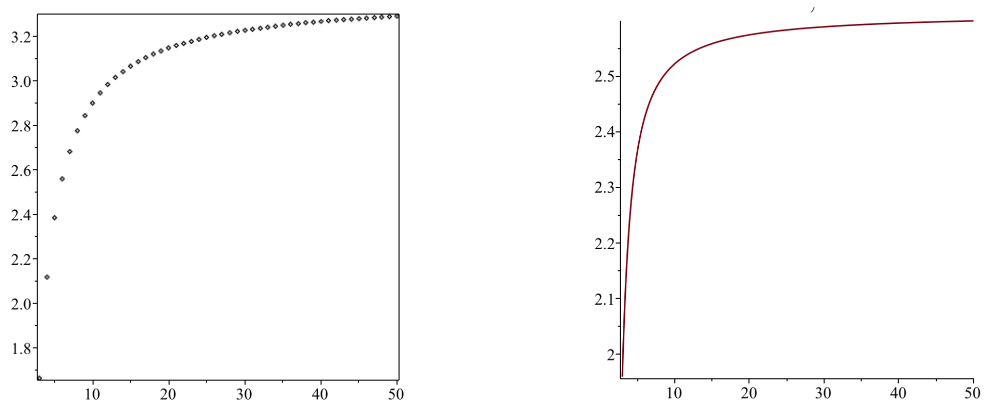
Let $H^\bullet(\mathrm{IF}(a,n)) = QH^*(\mathrm{IF}(a,n))|_{q=1}$ and $\hat{c}_1(\mathrm{IF}(a,n)) : H^\bullet(\mathrm{IF}(a,n)) \to H^\bullet(\mathrm{IF}(a,n))$ be defined by

$$y \in H^\bullet(\mathrm{IF}(a,n)) \mapsto \hat{c}_1(y) := (c_1(\mathrm{IF}(a,n)) \star y)|_{q=1}.$$

The matrix representation of linear operator $\hat{c}_1(F)$ are known to be irreducible when $F = G/P$ (e.g. see [CL17]).

Proposition 4.2. *Using* **numerical calculations** *we have that Galkin's Lower Bound Conjecture holds*

(1) *For numerous cases of partial flags* $\mathrm{IF}(a;n)$*;*
(2) *For* $\mathrm{OG}(m,2n+1)$ *where*
    (a) $m = 2, 2 \leqslant n \leqslant 50$*;*
    (b) $m = 3, 3 \leqslant n \leqslant 20$*;*
    (c) $m = 4, 4 \leqslant n \leqslant 10$*;*
(3) *For* $\mathrm{IG}(m,2n)$ *where*



(A) $\deg q \cdot \mathrm{FPdim}(\sigma_{(1)}) - \dim \mathrm{IG}(2, 2n) - 1$ (B) $\deg q \cdot f_k(n) - \dim \mathrm{IG}(2, 2n) - 1$

FIGURE 2. This is an example that the conjecture about the transcendental function given in subfigure 1b are coherent with the calculations for checking Galkin's Lower Bound Conjecture.

   (a) $m = 2, 2 \leqslant n \leqslant 50$;
   (b) $m = 3, 3 \leqslant n \leqslant 20$;
   (c) $m = 4, 4 \leqslant n \leqslant 10$;
(4) *For* $\mathrm{OG}(m, 2n + 2)$ *where*
   (a) $m = 2, 2 \leqslant n \leqslant 50$;
   (b) $m = 3, 3 \leqslant n \leqslant 20$;
   (c) $m = 4, 4 \leqslant n \leqslant 10$.
(5) *In Type* G_2*, the largest real eigenvalue of* $c_1(\mathrm{Fl}_G)$ *is approximately* $10.6012 > 7$.

The computations for partial flags were performed by a program written in Python. See [War]. The computations for isotropic Grassmannians in Types BCD where performed in a program written by Buch in Maple. See [Buc].

More can be said beyond the numerical calculations. Consider the function

$$R_m(n) := \mathrm{FPdim}(c_1(X)) - \dim X - 1$$

where $X \in \{\mathrm{OG}(m, 2n + 1), \mathrm{IG}(m, 2n), \mathrm{OG}(m, 2n + 2)\}$.

For notational ease we will use R instead of R_m. Our numerical results support the following conjecture:

Conjecture 5. *For any* $X \in \{\mathrm{OG}(m, 2n+1), \mathrm{IG}(m, 2n), \mathrm{OG}(m, 2n+2)\}$ *the function* $G(n)$ *extends to a second differentiable function* $R(x) : \mathbb{R}^+ \to \mathbb{R}$*;*

(1) *The limit* $\lim_{x\to\infty} R(x)$ *exists;*
(2) *The function* $R(x)$ *is concave down for* $x > M$ *for some* M*.*

See Figure 2a for an example.

REFERENCES

[BKT09] Anders Skovsted Buch, Andrew Kresch, and Harry Tamvakis. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math.*, 178(2):345–405, 2009.

[BM] Vladimiro Benedetti and Laurent Manivel. The small quantum cohomology of the cayley Grassmannian. Arxiv: 1907.07511v1.

[Buc] Anders Buch. Quantum calculator. https://sites.math.rutgers.edu/ asbuch/qcalc/.

[Buc05] Anders Skovsted Buch. Quantum cohomology of partial flag manifolds. *Trans. Amer. Math. Soc.*, 357(2):443–458, 2005.

[Buc15] Buch, Anders S. and Mihalcea, Leonardo C. Curve neighborhoods of Schubert varieties. *J. Differential Geom.*, 99(2):255–283, 2015.

[CH] Daewoong Cheong and Manwook Han. Galkin's lower bound conjecture for Lagrangian and orthogonal Grassmannians. ArXiv: 1906.11646v1.

[Che17] Daewoong Cheong. Quantum multiplication operators for Lagrangian and orthogonal Grassmannians. *J. Algebraic Combin.*, 45(4):1153–1171, 2017.

[CL17] Daewoong Cheong and Changzheng Li. On the conjecture $\mathcal{O}$ of GGI for G/P. *Adv. Math.*, 306:704–721, 2017.

[EGNO15] P Etingof, S Gelaki, D Nikshych, and V Ostrik. Tensor categories. *Mathematical Surveys and Monographs*, (205), 2015.

[ENO05] P Etingof, D Nikshych, and V Ostrik. On fusion categories. *Annals of Mathematics*, (162):581–642, 2005.

[ESS$^+$] La'Tier Evans, Lisa Schneider, Ryan M. Shifler, Laura Short, and Stephanie Warman. Galkin's lower bound conjecture holds for the Grassmannian. To appear comm. Alg.

[FK93] J. Fröhlich and T. Kerler. Quantum groups, quantum categories and quantum field theory. *Lecture Notes in Math.*, (1542):1–25, 1993.

[Gal] S. Galkin. The conifold point. arXiv:1404.7388.

[GGI16] Sergey Galkin, Vasily Golyshev, and Hiroshi Iritani. Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures. *Duke Math. J.*, 165(11):2005–2077, 2016.

[HKLY] Jianxun Hu, Hua-Zhong Ke, Changzheng Li, and Tuo Yang. Gamma conjecture I for del Pezzo surfaces. in preparation.

[Jon83] V. Jones. Index for subfactor. *Invent. Math.*, (72):1–25, 1983.

[Ke] Hua-Zhong Ke. On Conjecture $\mathcal{O}$ for projective complete intersections. available on arχiv:1809.10869.

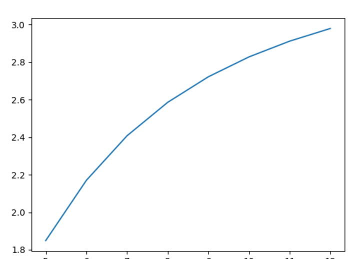
[LSYZ] Changzheng Li, Ryan M. Shifler, Mingzhi Yang, and Chi Zhang. On frobenius-perron dimension. Arxiv: 1909.01693v1.

[Rie01] Konstanze Rietsch. Quantum cohomology rings of Grassmannians and total positivity. *Duke Math. J.*, 110(3):523–553, 2001.

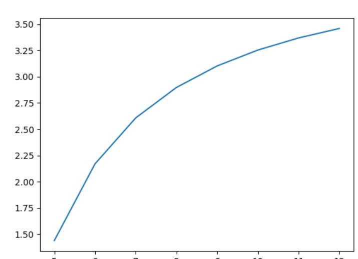
[War] Stephanie Warman. Galkin and Perron-Frobenius calculator for partial flags. https://github.com/swarman2/Galkin-and-Perron-Frobenius-Calculator-for-Parital-Flags/tree/master.

5. Appendix

Here we provide examples of plots of Frobenius-Perron Dimension (vertical axis) as n (horizontal axis) ranges for $\mathrm{Fl}(a;n), \mathrm{OG}(k,2n+1), \mathrm{IG}(k,2n), \mathrm{OG}(k,2n+2)$.

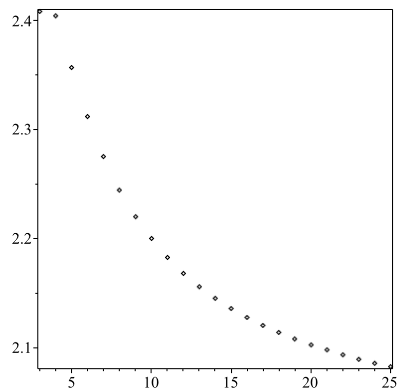


(A) $\mathrm{FPdim}(\sigma_{s_2})$ of $\mathrm{Fl}(2,4;n)$ for $5 \leqslant n \leqslant 12$.

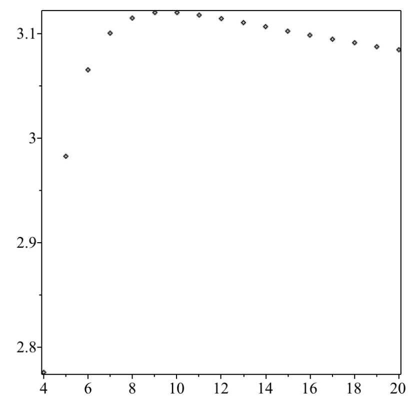


(B) $\mathrm{FPdim}(\sigma_{s_4})$ of $\mathrm{Fl}(2,4;n)$ for $5 \leqslant n \leqslant 12$.

Figure 3. Here we connect the points by a line to help with the visualization.

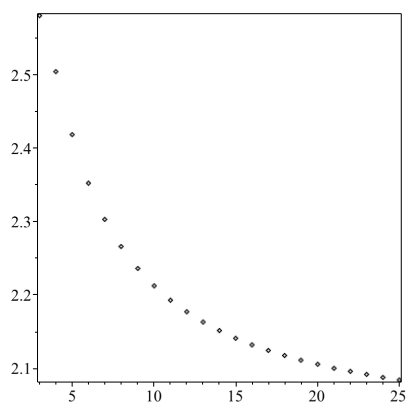


(A) FPdim($\sigma_{(2,1)}$) of IG$(2, 2n)$ for $3 \leqslant n \leqslant 25$.

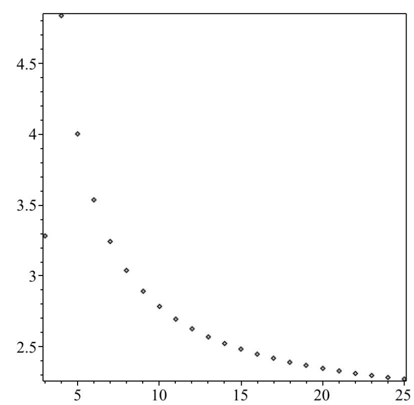


(B) FPdim($\sigma_{(1)}$) of IG$(3, 2n)$ for $4 \leqslant n \leqslant 25$.

FIGURE 4. IG$(m, 2n)$

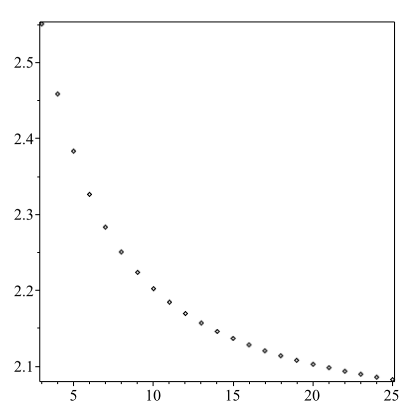


(A) FPdim($\sigma_{(1)}$) of OG$(2, 2n+1)$ for $3 \leqslant n \leqslant 25$.

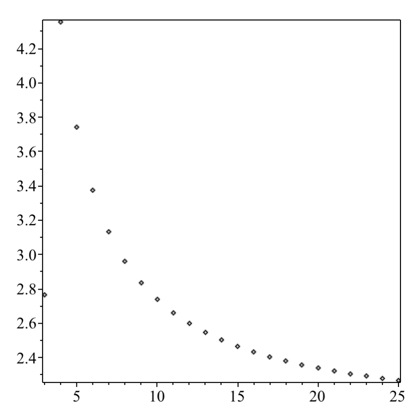


(B) FPdim($\sigma_{(2,1)}$) of OG$(2, 2n+1)$ for $3 \leqslant n \leqslant 25$.

FIGURE 5. OG$(2, 2n+1)$



(A) FPdim($\sigma_{(1)}$) of OG$(2, 2n+2)$ for $3 \leqslant n \leqslant 25$.



(B) FPdim($\sigma_{(2,1)}$) of OG$(2, 2n+2)$ for $3 \leqslant n \leqslant 25$.

FIGURE 6. OG$(2, 2n+2)$

Department of Mathematics and Computer Science, Henson Science Hall, Salisbury University, Salisbury MD 21801 USA
Email address: rmshifler@salisbury.edu

Department of Mathematics and Computer Science, Henson Science Hall, Salisbury University, Salisbury MD 21801 USA
Email address: swarman2@gulls.salisbury.edu