CONJECTURE $\mathcal{O}$ HOLDS FOR SOME HOROSPHERICAL VARIETIES OF PICARD RANK 1

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ABSTRACT. Property $\mathcal{O}$ for arbitrary complex, Fano manifolds X, is a statement about the eigenvalues of the linear operator obtained from the quantum multiplication of the anticanonical class of X. Pasquier listed the non-homogenous horospherical varieties of Picard rank 1 into five classes. Property $\mathcal{O}$ has already been shown to hold for the odd symplectic Grassmannian which is one class. We will show that Property $\mathcal{O}$ holds for two more classes and an example in a third class of Pasquier's list. The theory of Perron-Frobenius reduces our proofs to be graph theoretic.

1. Introduction

The purpose of this paper is to prove that Conjecture $\mathcal{O}$ holds for some horospherical varieties of Picard rank 1. We first consider a more tangible example to build intuition for the topic and the proof itself.

We call $\mathbb{P}^2$ the *projective plane*. It resembles $\mathbb{C}^2$ with an added property that any two distinct lines will intersect exactly once. The projective plane is defined as

$$\mathbb{P}^2 = \{[x;y;z] | x,y,z,\lambda \in \mathbb{C}, [x;y;z] = [\lambda x; \lambda y; \lambda z], \lambda \neq 0, x,y,z \text{ not all equal to } 0\}$$

Lines in $\mathbb{P}^2$ have the form $aX + bY + cZ = 0$. For the sake of clarity we will use the capitalized coordinates when considering the line in $\mathbb{P}^2$ and the lowercase coordinates when considering the same line in $\mathbb{C}^2$. Consider $a + by + cz = 0$ and $d + by + cz = 0$. These lines in $\mathbb{C}^2$ never intersect. However, if we make these lines homogeneous by rewriting them as $aX + bY + cZ = 0$ and $dX + bY + cZ = 0$ in $\mathbb{P}^2$, then we can recover the original lines that were parallel in $\mathbb{C}^2$ by setting $X = 1$. If we project these lines onto a different $\mathbb{C}^2$ by setting $Z = 1$ instead, we have the equations $ax + by + c = 0$ and $dx + by + c = 0$ which do intersect at the point $(0, -\frac{c}{b})$.

Similarly, $\mathbb{P}^1$ is called the projective line, and resembles $\mathbb{C}$. Lines in $\mathbb{P}^2$ look similar to $\mathbb{P}^1$. There is a natural sequence of embeddings: $\{pt\} \hookrightarrow \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, where pt is a point. The image of pt in $\mathbb{P}^2$ is pt and the image of $\mathbb{P}^1$ is a hyperplane hp (or line).

We consider the case $\mathbb{P}^2$, where $[pt]$, $[hp]$, and $[\mathbb{P}^2]$ means that we are considering pt, hp, and $\mathbb{P}^2$ as "generally" situated. We will be considering the intersection of these objects later on, so this throws out fringe cases we're not interested in. A generally situated point and line will not intersect, just as two generally situated lines will not overlap entirely. We also make use of Poincaré duals: $[\mathbb{P}^2]^\vee = [pt], [hp]^\vee = [hp], [pt]^\vee = [\mathbb{P}^2]$. An incomplete explanation (although sufficient for this example) of Poincaré duals would be "dimensional

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compliments". In a 2 dimensional space, the Poincaré dual of a 2 dimensional object would be a 0 dimensional object, while the Poincaré dual of a 1 dimensional object would remain a 1 dimensional object.

With this notation, $[pt], [hp]$, and $[\mathbb{P}^2]$ generate a commutative ring $\mathrm{QH}^*(\mathbb{P}^2)$ called the quantum cohomology. The operations are $\star$, which are the "intersections", and $+$ which is formal addition. "Intersections" is in quotes because it only partially describes the operation. For certain cases the intuition of intersections will lead to a correct answer. The intersection between two generally situated "lines" in $\mathbb{P}^2$ is a point, and $[hp] \star [hp] = [pt]$. The intersection of any object with the entirety of the space it belongs to will be the object itself, and $[\mathbb{P}^2] \star [hp] = [hp]$. This last equation also showcases why $[\mathbb{P}^2]$ is the identity of $\star$.

However, problems arise when we consider $[hp] \star [pt]$. Our intuitive understanding of $\star$ is insufficient in this case, as the intersection of a generally situated line and point would not exist. We need a more formal understanding of $\star$ to correct this. We reframe the earlier equation $[\mathbb{P}^2] \star [hp] = [hp]$ as $[\mathbb{P}^2] \star [hp] = 1q^0[hp]^\vee$ since there is exactly one point (i.e. a degree 0 curve) that intersects $\mathbb{P}^2$ and two general hyperplanes. We recall that $1q^0 = 1$, and that $[hp]^\vee = [hp]$, and so our earlier claim still holds true. However, this new understanding allows us to reevaluate $[hp] \star [pt]$. $[hp] \star [pt] = 1q^1[pt]^\vee$ since there is exactly one hyperplane (i.e. a degree 1 curve) that intersects a hyperplane and two points in general position.

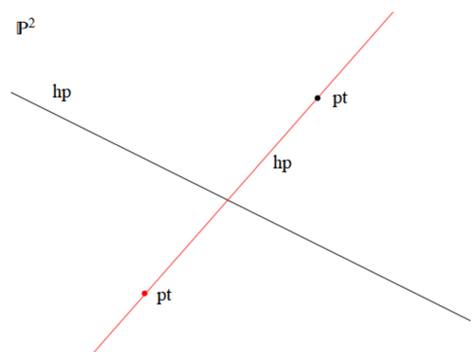


Figure 1. An example of $[hp] \star [pt] = 1q^1[pt]^\vee$

With this new understanding of $\star$ we are able to construct the entire multiplication table of $\mathrm{QH}^*(\mathbb{P}^2)$.

(1) $[\mathbb{P}^2] \star [\mathbb{P}^2] = 1q^0[pt]^\vee = [\mathbb{P}^2]$
(2) $[\mathbb{P}^2] \star [hp] = 1q^0[hp]^\vee = [hp]$
(3) $[\mathbb{P}^2] \star [pt] = 1q^0[\mathbb{P}^2]^\vee = [pt]$
(4) $[hp] \star [hp] = 1q^0[\mathbb{P}^2]^\vee = [pt]$
(5) $[hp] \star [pt] = 1q^1[pt]^\vee = 1q^1[\mathbb{P}^2]$
(6) $[pt] \star [pt] = 1q^1[hp]^\vee = 1q^1[hp]$

We use q to denote a quantum correction that algebraically accounts for the "fuzziness" of the intersections described by $\star$. If we consider $\deg q = 3$ for the case of $\mathbb{P}^2$, then the codimensions will add up in these equations as well.

We consider the linear operator $\hat{c_1}$ obtained from the multiplication of the anticanonical class $3[hp]$ and setting $q = 1$.

$$3[hp] \star [\mathbb{P}^2] = 3[hp]$$
$$3[hp] \star [hp] = 3[pt]$$
$$3[hp] \star [pt] = 3[\mathbb{P}^2]$$

$3[hp]\star$	$[pt]$	$[hp]$	$[\mathbb{P}^2]$
$[pt]$	0	0	3
$[hp]$	3	0	0
$[\mathbb{P}^2]$	0	3	0

The equations on the left are constructed from the multiplication table with $q = 1$. The matrix on the right is $\hat{c_1}$ and it has the characteristic polynomial $\lambda^3 - 27$. This means that the eigenvalues of $\hat{c_1}$ are three times the third roots of unity, $3, 3e^{\frac{2\pi}{3}i}, 3e^{\frac{4\pi}{3}i}$. The projective plane $\mathbb{P}^2$ is an example of a Fano variety, a specific type of smooth complex projective algebraic variety, for which Conjecture $\mathcal{O}$ holds.

We recall the precise statement of Conjecture $\mathcal{O}$, following [2, section 3]. Let F be a Fano variety, let $K := K_F$ be the canonical line bundle of F, let F_D be a fundamental divisor of F, and let $c_1(F) := c_1(-K) \in H^2(F)$ be the anitcanonical class. The Fano index of F is r, where r is the greatest integer such that $K_F \cong -rF_D$. The quantum cohomology ring $(QH^*(F), \star)$ is a graded algebra over $\mathbb{Z}[q]$, where q is the quantum parameter. Consider the specialization $H^\bullet(F) := QH^*(F)|_{q=1}$ at $q = 1$. The quantum multiplication by the first Chern class $c_1(F)$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $H^\bullet(F)$:

$$y \in H^\bullet(F) \mapsto \hat{c}_1(y) := (c_1(F) \star y)|_{q=1}.$$

Denote by $\delta_0 := \max\{|\delta| : \delta \text{ is an eigenvalue of } \hat{c}_1\}$. Then Property $\mathcal{O}$ states the following:

(1) The real number δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity one.
(2) If δ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0\gamma$ for some r-th root of unity $\gamma \in \mathbb{C}$, where r is the Fano index of F.

The property $\mathcal{O}$ was conjectured to hold for any Fano, complex manifold F by Galkin, Golyshev, and Iritani in [2]. If a Fano, complex, manifold has Property $\mathcal{O}$ then we say that the space satisfies Conjecture $\mathcal{O}$.

We note that the Fano index of our previous example, $\mathbb{P}^2$, is $r = 3$, and that $\delta_0 = 3$. δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity one, and every other eigenvalue of equal modulus is three times a third root of unity. So $\mathbb{P}^2$ satisfies Conjecture $\mathcal{O}$.

Next we recall the definition of a horospherical variety following [3]. Let G be a complex reductive group. A G-variety is a reduced scheme of finite type over the field of complex numbers $\mathbb{C}$, equipped with an algebraic action of G. Let B be a Borel subgroup of G. A G-variety X is called spherical if X has a dense B-orbit. Let X be a G-spherical variety and let H be the stabilizer of a point in the dense G-orbit in X. The variety X is called *horospherical* if H contains a conjugate of the maximal unipotent subgroup of G contained in the Borel subgroup B.

Horospherical varieties of Picard rank 1 were classified by Pasquier in [6]. The varieties are either homogeneous or can be constructed in a uniform way via a triple (Type(G),ω_Y,ω_Z) of representation-theoretic data, where Type(G) is the semisimple Lie type of the reductive group G and ω_Y, ω_Z are the fundamental weights. See [6, Section 1.3] for details. Pasquier classified the possible triples in five classes:

(1) $(B_n, \omega_{n-1}, \omega_n)$ with $n \geq 3$;
(2) $(B_3, \omega_1, \omega_3)$;
(3) $(C_n, \omega_m, \omega_{m-1})$ with $n \geq 2$ and $m \in [2, n]$;
(4) $(F_4, \omega_2, \omega_3)$;
(5) $(G_2, \omega_1, \omega_2)$.

In Propsition 3.6 of [7], Pasquier showed the triples in the above list are Fano varieties. Conjecture $\mathcal{O}$ has already been proved for the homogeneous case by Cheong and Li in [1] and for case (3), the odd symplectic Grassmannian, by Li, Mihalcea, and Shifler in [4]. We are now able to state the main theorem:

Theorem 1. If F belongs to the classes (1) for $n = 3$, (2), (3), and (5) of Pasquier's list, then Conjecture $\mathcal{O}$ holds for F.

2. Preliminaries

2.1. Quantum Cohomology. The small quantum cohomology is defined as follows. Let $(\alpha_i)_i$ be a basis of $H^*(F, \mathbb{R})$ and let $(\alpha_i^\vee)_i$ be the dual basis for the Poincaré pairing. The multiplication is given by

$$\alpha_i \star \alpha_j = \sum_{d \geq 0, k} c_{i,j}^{k,d} q^d \alpha_k$$

where $c_{i,j}^{k,d}$ are the 3-point, genus 0, Gromov-Witten invariants corresponding to rational curves of degree d intersecting the classes α_i, α_j, and $\alpha_k^\vee$. We will make use of the quantum Chevalley formula which is the multiplication of a hyperplane class hp with another class a_j. The result [3, Theorem 0.0.3] implies that if F belongs to the classes (1) for $n = 3$, (2), or (5) of Pasquier's list, then there is an explicit quantum Chevalley formula. The explicit quantum Chevalley formula is the key ingredient used to prove Property $\mathcal{O}$ holds.

2.2. Sufficient Criterion for Property $\mathcal{O}$ to hold. We recall the notion of the (oriented) quantum Chevalley Bruhat graph of a Fano variety F. The vertices of this graph are the basis elements $\alpha_i \in H^\bullet(F) := QH^*(F)|_{q=1}$. There is an oriented edge $\alpha_i \to \alpha_j$ if the class α_j appears with positive coeffiecient (we consider $q > 0$) in the quantum Chevalley multiplication $hp \star \alpha_i$ for some hyperplane class hp. The techniques involving Perron-Frobenius theory used by Li, Mihalcea, and Shifler in [4] and Cheong and Li in [1] imply the following lemma:

Lemma 1. If the following conditions hold for a Fano variety F:

(1) the matrix representation of $\hat{c}_1$ is nonnegative,
(2) the quantum Chevalley Bruhat graph of F is strongly connected, and
(3) there exists a cycle of length r, the Fano index, in the quantum Chevalley Bruhat graph of F,

then Property $\mathcal{O}$ holds for F. We will often refer to Lemma 1 specifically as "the lemma".

We refer the reader to [5, section 4.3] for further details on Perron-Frobenius theory. However, we provide here an explanation for why the lemma implies Conjecture $\mathcal{O}$.

Definition 1. The matrix M is *irreducible* in the sense that PMP^t is never of the form $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ for any permutation matrix P, where A, D are square submatrices.

Definition 2. The adjacency matrix $A(D)$ of a directed graph D with n vertices is the $(0,1)$-matrix whose (i,j) entry is 1 if and only if (i,j) is an arc of D. A directed graph $D(X)$ is said to be *associated* with a nonnegative matrix X, if the adjacency matrix of $D(X)$ has the same zero pattern as X.

Lemma 2. A nonnegative martix is irreducible if and only if the associated direct graph is strongly connected.

Lemma 3. An irreducible nonnegative matrix M has a real positive eigenvalue δ_0 such that $\delta_0 \geq |\delta|$ for any eigenvalue δ of M.

Lemma 4. Let M be an irreducible $n \times n$ matrix with maximal eigenvalue δ_0 and index r. If $\delta_1, \delta_2, ..., \delta_r$ are the eigenvalues of M with modulus δ_0, then $\delta_1, \delta_2, ..., \delta_r$ are equal to δ_0 times the distinct rth roots of unity.

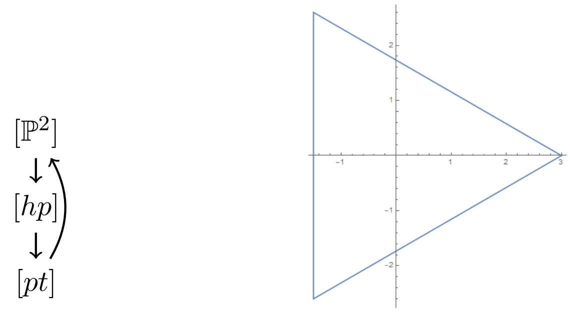
Lemma 5. The index of imprimitivity of an irreducible matrix is equal to the index of imprimitivity of the associated direct graph.

Then the lemma implies Conjecture $\mathcal{O}$ as follows:

(1) Note that by how the QCBG is defined, the QCBG is equivalent to the associated directed graph of the matrix representation of $\hat{c}_1$.
(2) The matrix representation of $\hat{c}_1$ is nonnegative and the quantum Chevalley Bruhat graph of F is strongly connected together implies that the matrix representation of $\hat{c}_1$ is irreducible by Lemma 2. This means that the matrix representation of $\hat{c}_1$ has a **real positive maximal eigenvalue** δ_0 by Lemma 3.
(3) Let M be a nonnegative matrix and D be the associated directed graph. The g.c.d. of the lengths of all cycles in D is called the *index of imprimitivity of* D, or simply the *index of* D. The index of M is the number of eigenvalues of M of modulus δ_0, where δ_0 is the maximal eigenvalue of M. We know by Lemma 5 that the index of D is equal to the index of M.
(4) It is a generally known fact that the Fano index r divides the index of D. However, as the index of D is the g.c.d. of all cycles in D, the index of D divides r so long as there is a cycle of length r in D. If there is a cycle in D of length r, then r is equal to the index of D and M.
(5) The existence of a cycle of length r, the Fano index, in the quantum Chevalley Bruhat graph of F implies that the index of the matrix representation of $\hat{c}_1$ is r. Therefore there are exactly r **eigenvalues** of the matrix representation of $\hat{c}_1$ with **modulus equal to** δ_0 such that these eigenvalues are δ_0 **times the distinct** r**th roots of unity** by Lemma 4.

Therefore if the conditions of the lemma hold for a Fano variety F, then F satisfies Conjecture $\mathcal{O}$. We observe that the conditions of the lemma hold for the example of $\mathbb{P}^2$.

$$\begin{bmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$



(a) Nonnegative **(b)** Strongly connected, $r = 3$ **(c)** Eigenvalues plotted on $\mathbb{C}$

We note that while the lemma implies Conjecture $\mathcal{O}$, this is not an iff implication. There exists a Fano variety where Conjecture $\mathcal{O}$ holds but the lemma from before does not apply. In this case Withrow (2018) calculated the matrix $\hat{c}_1$ to be

$$\begin{bmatrix} 0 & 2 & 2 & -2 & 0 & 0 & 3 & 4 \\ 3 & -1 & 2 & 1 & 2 & 4 & 0 & 3 \\ 1 & 1 & -1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 & -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 3 & 2 & 0 \end{bmatrix}$$

The lemma is generalized in a recent paper by Hu, Ke, Li, and Yang [8] to include some matrices with negative entries.

3. Checking Property $\mathcal{O}$ Holds

Let X be a horospherical variety. We will simplify our notation where the basis of $H^\bullet(X)$ is $\{\iota, hp, \alpha_i\}_{i\in I}$ for some finite index set I. Observe by [3] that the anticanonical classes are

$$c_1(X) = \begin{cases} 5[hp] & \text{when X is case (1) for } n = 3 \\ 7[hp] & \text{when X is case (2)} \\ 4[hp] & \text{when X is case (5)} \end{cases}$$

and the Fano indices are

$$r = \begin{cases} 5 & \text{when X is case (1) for } n = 3 \\ 7 & \text{when X is case (2)} \\ 4 & \text{when X is case (5)} \end{cases}.$$

The endomorphism $\hat{c}_1$ acting on the basis elements of $H^\bullet(X)$ are determined by the Chevalley formula in the following way:

$$\begin{aligned} \hat{c}_1(\alpha_i) &= 5(hp \star \alpha_i)|_{q=1} \text{ when X is case (1) for } n = 3, \\ \hat{c}_1(\alpha_i) &= 7(hp \star \alpha_i)|_{q=1} \text{ when X is case (2), and} \\ \hat{c}_1(\alpha_i) &= 4(hp \star \alpha_i)|_{q=1} \text{ when X is case (5).} \end{aligned}$$

Each of the following three subsections will show that Conjecture $\mathcal{O}$ holds for case (1) for $n = 3$, case (2), and case (5) of Pasquier's list, respectively. In each subsection we will reformulate the quantum Chevalley formulas stated in [3], present the quantum Chevalley Bruhat graph, and argue that each condition of 1 is satisfied. For each case, we have kept the same format of the equations presented by Pech et al. with our prescribed basis for ease of identification for the reader.

3.1. Case (1) for $n = 3$. We will reformulate the quantum Chevalley formula stated in [3] using the basis

$$G := \{\iota, [hp], \alpha_1, \alpha_2, \cdots, \alpha_{18}\}.$$

Proposition 1. *The following equalities hold by* [3, Proposition 4.2.1].

$\hat{c}_1(\iota) = 5[hp]$
$\hat{c}_1(hp) = 10\alpha_1 + 5\alpha_2$
$\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4$
$\hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5$
$\hat{c}_1(\alpha_3) = 10\alpha_6 + 5\alpha_7 + 5\alpha_8$
$\hat{c}_1(\alpha_4) = 5\alpha_6 + 10\alpha_7$
$\hat{c}_1(\alpha_5) = 5\alpha_8$
$\hat{c}_1(\alpha_6) = 10\alpha_9 + 5\alpha_{10} + 5\alpha_{11}$
$\hat{c}_1(\alpha_7) = 5\alpha_{10}$
$\hat{c}_1(\alpha_8) = 5\alpha_{11} + 5\iota$
$\hat{c}_1(\alpha_9) = 5\alpha_{12} + 5\alpha_{13}$
$\hat{c}_1(\alpha_{10}) = 10\alpha_{13} + 5\alpha_{14}$
$\hat{c}_1(\alpha_{11}) = 5\alpha_{12} + 5\alpha_{14} + 5[hp]$
$\hat{c}_1(\alpha_{12}) = 5\alpha_{15} + 5\alpha_1$
$\hat{c}_1(\alpha_{13}) = 5\alpha_{15} + 5\alpha_{16}$
$\hat{c}_1(\alpha_{14}) = 5\alpha_{15} + 5\alpha_2$
$\hat{c}_1(\alpha_{15}) = 5\alpha_{17} + 5\alpha_3$
$\hat{c}_1(\alpha_{16}) = 5\alpha_{17} + 5\alpha_5$
$\hat{c}_1(\alpha_{17}) = 5\alpha_{18} + 5\alpha_6 + 5\alpha_8$
$\hat{c}_1(\alpha_{18}) = 5\alpha_9 + 5\alpha_{11} + 10\iota$

The following is the quantum Chevalley Bruhat graph of the Fano variety X in case (1) for $n = 3$.

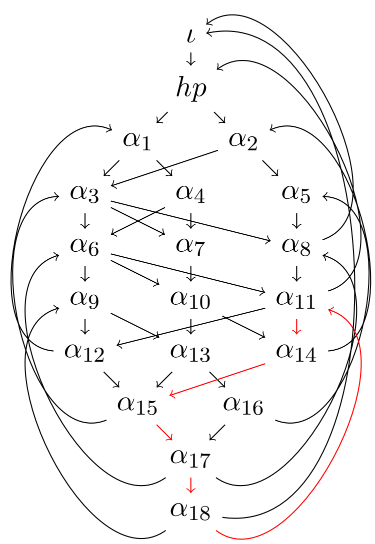
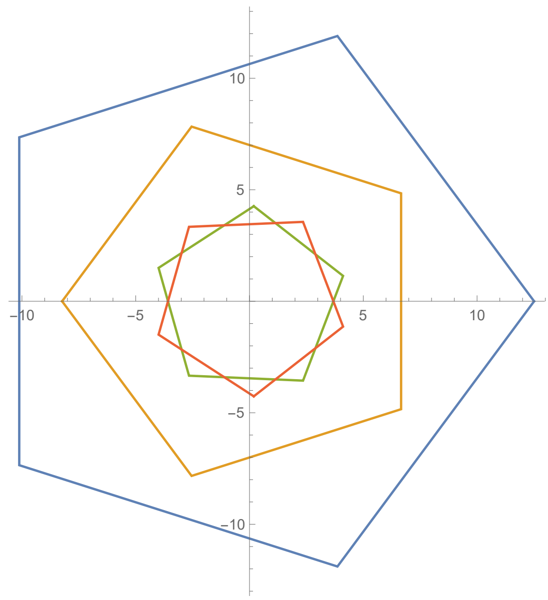


Figure 3. Class (1) QCBG: strongly connected, cycle of length $r = 5$.

Lemma 6. Property $\mathcal{O}$ holds when X is case (1) with $n = 3$ of Pasquier's list.

Proof. The coefficients that appear in the equations in Proposition 1 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 3, and the cycle $\alpha_{18}\alpha_{11}\alpha_{14}\alpha_{15}\alpha_{17}\alpha_{18}$ has length $r = 5$. □

This graph is a geometric representation of Property $\mathcal{O}$. The eigenvalues of $\hat{c}_1(\alpha_i) = 5(hp \star \alpha_i)|_{q=1}$ are plotted on $\mathbb{C}$ and then eigenvalues of equal modulus are connected by lines that form regular polygons.



3.2. Case (2). Again, we reformulate the quantum Chavelley formula from [3] using the basis

$$G := \{\iota, [hp], \alpha_1, \alpha_2, \cdots, \alpha_{12}\}.$$

Proposition 2. *The following equalities hold by* [3, Proposition 4.3.1].

$\hat{c}_1(\iota) = 7[hp]$
$\hat{c}_1(hp) = 7\alpha_1$
$\hat{c}_1(\alpha_1) = 14\alpha_2 + 7\alpha_3$
$\hat{c}_1(\alpha_2) = 7\alpha_4 + 7\alpha_5$
$\hat{c}_1(\alpha_3) = 7\alpha_5$
$\hat{c}_1(\alpha_4) = 7\alpha_6 + 7\alpha_7$
$\hat{c}_1(\alpha_5) = 7\alpha_7$
$\hat{c}_1(\alpha_6) = 7\alpha_8$
$\hat{c}_1(\alpha_7) = 7\alpha_8 + 7\alpha_9$
$\hat{c}_1(\alpha_8) = 7\alpha_{10}$
$\hat{c}_1(\alpha_9) = 7\alpha_{10} + 7\iota$
$\hat{c}_1(\alpha_{10}) = 7\alpha_{11} + 7[hp]$
$\hat{c}_1(\alpha_{11}) = 7\alpha_{12} + 7\alpha_1$
$\hat{c}_1(\alpha_{12}) = 7\alpha_2$

The associated quantum Chevalley Bruhat graph is

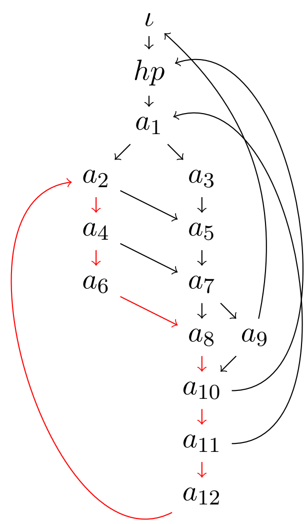
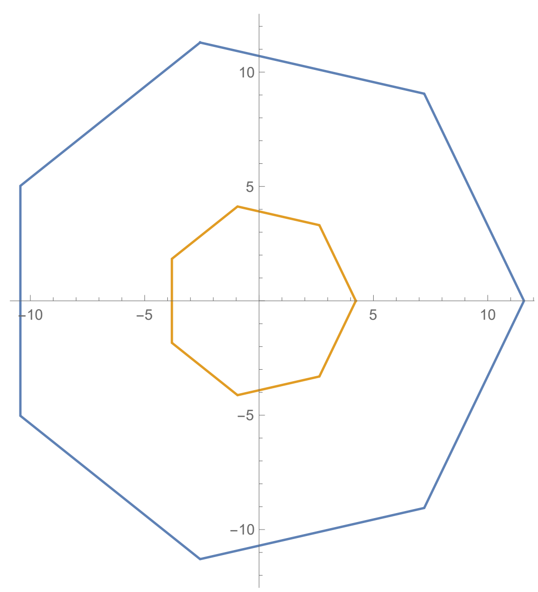


Figure 5. Class (2) QCBG: Strongly connected, cycle of length $r = 7$.

Lemma 7. Property $\mathcal{O}$ holds when X is case (2) of Pasquier's list.

Proof. The coefficients that appear in the equations in Proposition 2 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 5, and the cycle $\alpha_{12}\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10}\alpha_{11}\alpha_{12}$ has length $r = 7$. □

This graph is a geometric representation of Property $\mathcal{O}$. The eigenvalues of $\hat{c}_1(\alpha_i) = 7(hp \star \alpha_i)|_{q=1}$ are plotted on $\mathbb{C}$ and then eigenvalues of equal modulus are connected by lines that form regular polygons.



3.3. Case(5). Again, we reformulate the quantum Chavelley formula from [3] using the basis

$$G := \{\iota, [hp], \alpha_1, \alpha_2, \cdots, \alpha_{10}\}.$$

Proposition 3. *The following equalities hold by* [3, Proposition 4.5.1].

$\hat{c}_1(\iota) = 4[hp]$
$\hat{c}_1(hp) = 12\alpha_1 + 4\alpha_2$
$\hat{c}_1(\alpha_1) = 8\alpha_3 + 4\alpha_4$
$\hat{c}_1(\alpha_2) = 4\alpha_4$
$\hat{c}_1(\alpha_3) = 12\alpha_5 + 4\alpha_6$
$\hat{c}_1(\alpha_4) = 4\alpha_6 + 4\iota$
$\hat{c}_1(\alpha_5) = 4\alpha_7 + 4\alpha_8$
$\hat{c}_1(\alpha_6) = 8\alpha_7 + 4[hp]$
$\hat{c}_1(\alpha_7) = 4\alpha_9 + 4\alpha_1$
$\hat{c}_1(\alpha_8) = 4\alpha_9 + 4\alpha_2$
$\hat{c}_1(\alpha_9) = 4\alpha_{10} + 4\alpha_3 + 4\alpha_4$
$\hat{c}_1(\alpha_{10}) = 4\alpha_5 + 4\alpha_6 + 8\iota$

The associated quantum Chevalley Bruhat graph is

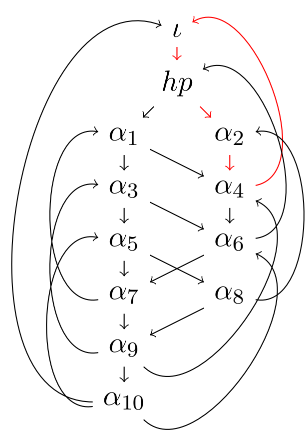
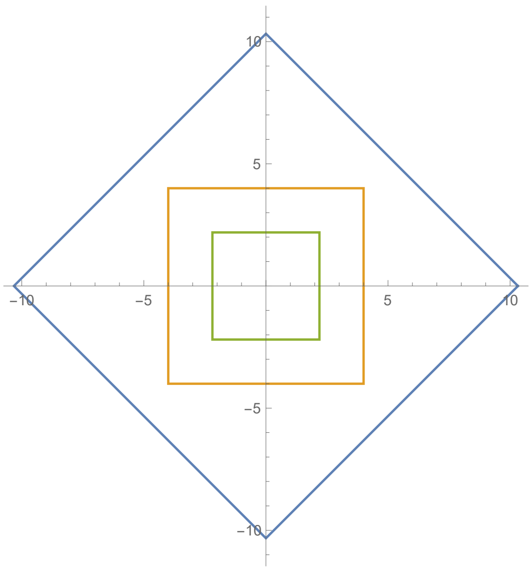


Figure 7. Class (5) QCBG: Strongly connected, cycle of length $r = 4$.

Lemma 8. Property $\mathcal{O}$ holds when X is case (5) of Pasquier's list.

Proof. The coefficients that appear in the equations in Proposition 3 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 7, and the cycle $\alpha_{10}\alpha_6\alpha_7\alpha_9\alpha_{10}$ has length $r = 4$. □

This graph is a geometric representation of Property $\mathcal{O}$. The eigenvalues of $\hat{c}_1(\alpha_i) = 4(hp \star \alpha_i)|_{q=1}$ are plotted on $\mathbb{C}$ and then eigenvalues of equal modulus are connected by lines that form regular polygons.



Theorem 1 follows from Lemmas 6, 7, and 8.

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