CONJECTURE \( O \) HOLDS FOR SOME HOROSPHERICAL VARIETIES OF PICARD RANK 1

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Abstract. Property \( O \) for arbitrary complex, Fano manifolds \( X \), is a statement about the eigenvalues of the linear operator obtained from the quantum multiplication of the anticanonical class of \( X \). Pasquier listed the non-homogenous horospherical varieties of Picard rank 1 into five classes. Property \( O \) has already been shown to hold for the odd symplectic Grassmannian which is one class. We will show that Property \( O \) holds for two more classes and an example in a third class of Pasquier’s list. The theory of Perron-Frobenius reduces our proofs to be graph theoretic.

1. Introduction

The purpose of this paper is to prove that Conjecture \( O \) holds for some horospherical varieties of Picard rank 1. We first consider a more tangible example to build intuition for the topic and the proof itself.

We call \( \mathbb{P}^2 \) the projective plane. It resembles \( \mathbb{C}^2 \) with an added property that any two distinct lines will intersect exactly once. The projective plane is defined as

\[
\mathbb{P}^2 = \left\{ [x; y; z] | x, y, z, \lambda \in \mathbb{C}, [x; y; z] = [\lambda x; \lambda y; \lambda z], \lambda \neq 0, x, y, z \text{ not all equal to 0} \right\}
\]

Lines in \( \mathbb{P}^2 \) have the form \( aX + bY + cZ = 0 \). For the sake of clarity we will use the capitalized coordinates when considering the line in \( \mathbb{P}^2 \) and the lowercase coordinates when considering the same line in \( \mathbb{C}^2 \). Consider \( a + by + cz = 0 \) and \( d + by + cz = 0 \). These lines in \( \mathbb{C}^2 \) never intersect. However, if we make these lines homogeneous by rewriting them as \( aX + bY + cZ = 0 \) and \( dX + bY + cZ = 0 \) in \( \mathbb{P}^2 \), then we can recover the original lines that were parallel in \( \mathbb{C}^2 \) by setting \( X = 1 \). If we project these lines onto a different \( \mathbb{C}^2 \) by setting \( Z = 1 \) instead, we have the equations \( ax + by + c = 0 \) and \( dx + by + c = 0 \) which do intersect at the point \((0, -\frac{c}{b})\).

Similarly, \( \mathbb{P}^1 \) is called the projective line, and resembles \( \mathbb{C} \). Lines in \( \mathbb{P}^2 \) look similar to \( \mathbb{P}^1 \). There is a natural sequence of embeddings: \( \{pt\} \hookrightarrow \mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \), where \( pt \) is a point. The image of \( pt \) in \( \mathbb{P}^2 \) is \( pt \) and the image of \( \mathbb{P}^1 \) is a hyperplane \( hp \) (or line).

We consider the case \( \mathbb{P}^2 \), where \([pt],[hp] \), and \([\mathbb{P}^2] \) means that we are considering \( pt, hp \), and \( \mathbb{P}^2 \) as “generally” situated. We will be considering the intersection of these objects later on, so this throws out fringe cases we’re not interested in. A generally situated point and line will not intersect, just as two generally situated lines will not overlap entirely. We also make use of Poincaré duals: \([\mathbb{P}^2] = [pt], [hp] = [hp], [pt] = [\mathbb{P}^2] \). An incomplete explanation (although sufficient for this example) of Poincaré duals would be “dimensional.

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compliments”. In a 2 dimensional space, the Poincaré dual of a 2 dimensional object would be a 0 dimensional object, while the Poincaré dual of a 1 dimensional object would remain a 1 dimensional object.

With this notation, \([pt], [hp],\) and \([\mathbb{P}^2]\) generate a commutative ring \(QH^*(\mathbb{P}^2)\) called the quantum cohomology. The operations are \(\star\), which are the “intersections”, and \(+\) which is formal addition. “Intersections” is in quotes because it only partially describes the operation. For certain cases the intuition of intersections will lead to a correct answer. The intersection between two generally situated “lines” in \(\mathbb{P}^2\) is a point, and \([hp] \star [hp] = [pt]\). The intersection of any object with the entirety of the space it belongs to will be the object itself, and \([\mathbb{P}^2] \star [hp] = [hp]\). This last equation also showcases why \([\mathbb{P}^2]\) is the identity of \(\star\).

However, problems arise when we consider \([hp] \star [pt]\). Our intuitive understanding of \(\star\) is insufficient in this case, as the intersection of a generally situated line and point would not exist. We need a more formal understanding of \(\star\) to correct this. We reframe the earlier equation \([\mathbb{P}^2] \star [hp] = [hp]\) as \([\mathbb{P}^2] \star [hp] = 1q^0[hp]^\vee\) since there is exactly one point (i.e. a degree 0 curve) that intersects \(\mathbb{P}^2\) and two general hyperplanes. We recall that \(1q^0 = 1\), and that \([hp]^\vee = [hp]\), and so our earlier claim still holds true. However, this new understanding allows us to reevaluate \([hp] \star [pt]\). \([hp] \star [pt] = 1q^1[pt]^\vee\) since there is exactly one hyperplane (i.e. a degree 1 curve) that intersects a hyperplane and two points in general position.

\[\begin{align*}
(1) \quad [\mathbb{P}^2] \star [\mathbb{P}^2] &= 1q^0[pt]^\vee = [\mathbb{P}^2] \\
(2) \quad [\mathbb{P}^2] \star [hp] &= 1q^0[hp]^\vee = [hp] \\
(3) \quad [\mathbb{P}^2] \star [pt] &= 1q^0[\mathbb{P}^2]^\vee = [pt] \\
(4) \quad [hp] \star [hp] &= 1q^0[\mathbb{P}^2]^\vee = [pt] \\
(5) \quad [hp] \star [pt] &= 1q^1[pt]^\vee = 1q^1[\mathbb{P}^2] \\
(6) \quad [pt] \star [pt] &= 1q^1[hp]^\vee = 1q^1[hp]
\end{align*}\]

We use \(q\) to denote a quantum correction that algebraically accounts for the “fuzziness” of the intersections described by \(\star\). If we consider \(\text{deg } q = 3\) for the case of \(\mathbb{P}^2\), then the codimensions will add up in these equations as well.
We consider the linear operator $\hat{c}_1$ obtained from the multiplication of the anticanonical class $3[hp]$ and setting $q = 1$.

$$
\begin{align*}
3[hp] \star [\mathbb{P}^2] &= 3[hp] \\
3[hp] \star [hp] &= 3[pt] \\
3[hp] \star [pt] &= 3[\mathbb{P}^2]
\end{align*}
$$

The matrix on the right is $\hat{c}_1$ and it has the characteristic polynomial $\lambda^3 - 27$. This means that the eigenvalues of $\hat{c}_1$ are three times the third roots of unity, $3, 3e^{\frac{2\pi}{3}i}, 3e^{\frac{4\pi}{3}i}$. The projective plane $\mathbb{P}^2$ is an example of a Fano variety, a specific type of smooth complex projective algebraic variety, for which Conjecture $O$ holds.

We recall the precise statement of Conjecture $O$, following [2] section 3. Let $F$ be a Fano variety, let $K := K_F$ be the canonical line bundle of $F$, let $F_D$ be a fundamental divisor of $F$, and let $c_1(F) := c_1(-K) \in H^2(F)$ be the anticanonical class. The Fano index of $F$ is $r$, where $r$ is the greatest integer such that $K_F \cong -rF_D$. The quantum cohomology ring $(\mathcal{Q}H^*(F), \star)$ is a graded algebra over $\mathbb{Z}[q]$, where $q$ is the quantum parameter. Consider the specialization $\mathcal{H}^*(F) := \mathcal{Q}H^*(F)|_{q=1}$ at $q = 1$. The quantum multiplication by the first Chern class $c_1(F)$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $\mathcal{H}^*(F)$:

$$y \in \mathcal{H}^*(F) \mapsto \hat{c}_1(y) := (c_1(F) \star y)|_{q=1}.$$  

Denote by $\delta_0 := \max\{|\delta| : \delta$ is an eigenvalue of $\hat{c}_1\}$. Then Property $O$ states the following:

1. The real number $\delta_0$ is an eigenvalue of $\hat{c}_1$ of multiplicity one.
2. If $\delta$ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0 \gamma$ for some $r$-th root of unity $\gamma \in \mathbb{C}$, where $r$ is the Fano index of $F$.

The property $O$ was conjectured to hold for any Fano, complex manifold $F$ by Galkin, Golyshev, and Iritani in [2]. If a Fano, complex, manifold has Property $O$ then we say that the space satisfies Conjecture $O$.

We note that the Fano index of our previous example, $\mathbb{P}^2$, is $r = 3$, and that $\delta_0 = 3$. $\delta_0$ is an eigenvalue of $\hat{c}_1$ of multiplicity one, and every other eigenvalue of equal modulus is three times a third root of unity. So $\mathbb{P}^2$ satisfies Conjecture $O$.

Next we recall the definition of a horospherical variety following [3]. Let $G$ be a complex reductive group. A $G$-variety is a reduced scheme of finite type over the field of complex numbers $\mathbb{C}$, equipped with an algebraic action of $G$. Let $B$ be a Borel subgroup of $G$. A $G$-variety $X$ is called spherical if $X$ has a dense $B$-orbit. Let $X$ be a $G$-spherical variety and let $H$ be the stabilizer of a point in the dense $G$-orbit in $X$. The variety $X$ is called horospherical if $H$ contains a conjugate of the maximal unipotent subgroup of $G$ contained in the Borel subgroup $B$.

Horospherical varieties of Picard rank 1 were classified by Pasquier in [6]. The varieties are either homogeneous or can be constructed in a uniform way via a triple $(\text{Type}(G), \omega_Y, \omega_Z)$ of representation-theoretic data, where Type($G$) is the semisimple Lie type of the reductive
group \( G \) and \( \omega_Y, \omega_Z \) are the fundamental weights. See [6, Section 1.3] for details. Pasquier classified the possible triples in five classes:

1. \((B_n, \omega_{n-1}, \omega_n)\) with \( n \geq 3 \);
2. \((B_3, \omega_1, \omega_3)\);
3. \((C_n, \omega_m, \omega_{m-1})\) with \( n \geq 2 \) and \( m \in [2, n] \);
4. \((F_4, \omega_2, \omega_3)\);
5. \((G_2, \omega_1, \omega_2)\).

In Proposition 3.6 of [7], Pasquier showed the triples in the above list are Fano varieties. Conjecture \( \mathcal{O} \) has already been proved for the homogeneous case by Cheong and Li in [1] and for case (3), the odd symplectic Grassmannian, by Li, Mihalcea, and Shifler in [4]. We are now able to state the main theorem:

**Theorem 1.** If \( F \) belongs to the classes (1) for \( n = 3 \), (2), (3), and (5) of Pasquier’s list, then Conjecture \( \mathcal{O} \) holds for \( F \).

2. Preliminaries

2.1. Quantum Cohomology. The small quantum cohomology is defined as follows. Let \((\alpha_i)_i \) be a basis of \( H^\bullet(F, \mathbb{R}) \) and let \((\alpha_i^\vee)_i \) be the dual basis for the Poincaré pairing. The multiplication is given by

\[
\alpha_i \star \alpha_j = \sum_{d \geq 0, k} c_{i,j}^{k,d} q^d \alpha_k
\]

where \( c_{i,j}^{k,d} \) are the 3-point, genus 0, Gromov-Witten invariants corresponding to rational curves of degree \( d \) intersecting the classes \( \alpha_i, \alpha_j, \) and \( \alpha_i^\vee \). We will make use of the quantum Chevalley formula which is the multiplication of a hyperplane class \( hp \) with another class \( a_j \). The result [3, Theorem 0.0.3] implies that if \( F \) belongs to the classes (1) for \( n = 3 \), (2), or (5) of Pasquier’s list, then there is an explicit quantum Chevalley formula. The explicit quantum Chevalley formula is the key ingredient used to prove Property \( \mathcal{O} \) holds.

2.2. Sufficient Criterion for Property \( \mathcal{O} \) to hold. We recall the notion of the (oriented) quantum Chevalley Bruhat graph of a Fano variety \( F \). The vertices of this graph are the basis elements \( \alpha_i \in H^\bullet(F) := QH^\bullet(F)_{q=1} \). There is an oriented edge \( \alpha_i \to \alpha_j \) if the class \( \alpha_j \) appears with positive coefficient (we consider \( q > 0 \)) in the quantum Chevalley multiplication \( hp \star \alpha_i \) for some hyperplane class \( hp \). The techniques involving Perron-Frobenius theory used by Li, Mihalcea, and Shifler in [4] and Cheong and Li in [11] imply the following lemma:

**Lemma 1.** If the following conditions hold for a Fano variety \( F \):

1. the matrix representation of \( c_1 \) is nonnegative,
2. the quantum Chevalley Bruhat graph of \( F \) is strongly connected, and
3. there exists a cycle of length \( r \), the Fano index, in the quantum Chevalley Bruhat graph of \( F \),

then Property \( \mathcal{O} \) holds for \( F \). We will often refer to Lemma 1 specifically as “the lemma”.
We refer the reader to [5, section 4.3] for further details on Perron-Frobenius theory. However, we provide here an explanation for why the lemma implies Conjecture $O$.

**Definition 1.** The matrix $M$ is irreducible in the sense that $PMP^t$ is never of the form
\[
\begin{bmatrix}
B & C \\
0 & D
\end{bmatrix}
\] for any permutation matrix $P$, where $A, D$ are square submatrices.

**Definition 2.** The adjacency matrix $A(D)$ of a directed graph $D$ with $n$ vertices is the $(0,1)$-matrix whose $(i,j)$ entry is 1 if and only if $(i,j)$ is an arc of $D$. A directed graph $D(X)$ is said to be associated with a nonnegative matrix $X$, if the adjacency matrix of $D(X)$ has the same zero pattern as $X$.

**Lemma 2.** A nonnegative martix is irreducible if and only if the associated direct graph is strongly connected.

**Lemma 3.** An irreducible nonnegative matrix $M$ has a real positive eigenvalue $\delta_0$ such that $\delta_0 \geq |\delta|$ for any eigenvalue $\delta$ of $M$.

**Lemma 4.** Let $M$ be an irreducible $n \times n$ matrix with maximal eigenvalue $\delta_0$ and index $r$. If $\delta_1, \delta_2, ..., \delta_r$ are the eigenvalues of $M$ with modulus $\delta_0$, then $\delta_1, \delta_2, ..., \delta_r$ are equal to $\delta_0$ times the distinct $r$th roots of unity.

**Lemma 5.** The index of imprimitivity of an irreducible matrix is equal to the index of imprimitivity of the associated direct graph.

Then the lemma implies Conjecture $O$ as follows:

1. Note that by how the QCBG is defined, the QCBG is equivalent to the associated directed graph of the matrix representation of $\hat{c}_1$.
2. The matrix representation of $\hat{c}_1$ is nonnegative and the quantum Chevalley Bruhat graph of $F$ is strongly connected together implies that the matrix representation of $\hat{c}_1$ is irreducible by Lemma 2. This means that the matrix representation of $\hat{c}_1$ has a real positive maximal eigenvalue $\delta_0$ by Lemma 3.
3. Let $M$ be a nonnegative matrix and $D$ be the associated directed graph. The g.c.d. of the lengths of all cycles in $D$ is called the index of imprimitivity of $D$, or simply the index of $D$. The index of $M$ is the number of eigenvalues of $M$ of modulus $\delta_0$, where $\delta_0$ is the maximal eigenvalue of $M$. We know by Lemma 5 that the index of $D$ is equal to the index of $M$.
4. It is a generally known fact that the Fano index $r$ divides the index of $D$. However, as the index of $D$ is the g.c.d. of all cycles in $D$, the index of $D$ divides $r$ so long as there is a cycle of length $r$ in $D$. If there is a cycle in $D$ of length $r$, then $r$ is equal to the index of $D$ and $M$.
5. The existence of a cycle of length $r$, the Fano index, in the quantum Chevalley Bruhat graph of $F$ implies that the index of the matrix representation of $\hat{c}_1$ is $r$. Therefore there are exactly $r$ eigenvalues of the matrix representation of $\hat{c}_1$ with modulus equal to $\delta_0$ such that these eigenvalues are $\delta_0$ times the distinct $r$th roots of unity by Lemma 4.

Therefore if the conditions of the lemma hold for a Fano variety $F$, then $F$ satisfies Conjecture $O$. We observe that the conditions of the lemma hold for the example of $\mathbb{P}^2$. 

\[
\begin{bmatrix}
0 & 0 & 3 \\
3 & 0 & 0 \\
0 & 0 & 3 \\
\end{bmatrix}
\]
\[\begin{bmatrix}
[\mathbb{P}^2] \\
[h_p] \\
[pt] \\
\end{bmatrix}
\]
(a) Nonnegative \hspace{1cm} (b) Strongly connected, \( r = 3 \) \hspace{1cm} (c) Eigenvalues plotted on \( \mathbb{C} \)

We note that while the lemma implies Conjecture \( \mathcal{O} \), this is not an iff implication. There exists a Fano variety where Conjecture \( \mathcal{O} \) holds but the lemma from before does not apply. In this case Withrow (2018) calculated the matrix \( \hat{c}_1 \) to be

\[
\begin{bmatrix}
0 & 2 & 2 & -2 & 0 & 0 & 3 & 4 \\
3 & -1 & 2 & 1 & 2 & 4 & 0 & 3 \\
1 & 1 & -1 & -1 & 0 & 0 & 2 & 0 \\
2 & 0 & 1 & 0 & 2 & 2 & 0 & 0 \\
0 & 1 & 4 & 0 & -1 & 1 & 0 & 2 \\
0 & 2 & 0 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 2 & 3 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 3 & 2 & 0 \\
\end{bmatrix}
\]

The lemma is generalized in a recent paper by Hu, Ke, Li, and Yang [8] to include some matrices with negative entries.

3. Checking Property \( \mathcal{O} \) Holds

Let \( X \) be a horospherical variety. We will simplify our notation where the basis of \( H^\bullet(X) \) is \( \{t, h_p, \alpha_i\}_{i \in I} \) for some finite index set \( I \). Observe by [3] that the anticanonical classes are

\[
c_1(X) = \begin{cases} 
5[h_p] & \text{when } X \text{ is case (1) for } n = 3 \\
7[h_p] & \text{when } X \text{ is case (2)} \\
4[h_p] & \text{when } X \text{ is case (5)} 
\end{cases}
\]

and the Fano indices are

\[
r = \begin{cases} 
5 & \text{when } X \text{ is case (1) for } n = 3 \\
7 & \text{when } X \text{ is case (2)} \\
4 & \text{when } X \text{ is case (5)} 
\end{cases}
\]

The endomorphism \( \hat{c}_1 \) acting on the basis elements of \( H^\bullet(X) \) are determined by the Chevalley formula in the following way:

\[
\hat{c}_1(\alpha_i) = 5(h_p \star \alpha_i)|_{q=1} \text{ when } X \text{ is case (1) for } n = 3, \\
\hat{c}_1(\alpha_i) = 7(h_p \star \alpha_i)|_{q=1} \text{ when } X \text{ is case (2)}, \text{ and} \\
\hat{c}_1(\alpha_i) = 4(h_p \star \alpha_i)|_{q=1} \text{ when } X \text{ is case (5)}. 
\]
Each of the following three subsections will show that Conjecture \( \mathcal{O} \) holds for case (1) for \( n = 3 \), case (2), and case (5) of Pasquier’s list, respectively. In each subsection we will reformulate the quantum Chevalley formulas stated in [3], present the quantum Chevalley Bruhat graph, and argue that each condition of [1] is satisfied. For each case, we have kept the same format of the equations presented by Pech et al. with our prescribed basis for ease of identification for the reader.

### 3.1. Case (1) for \( n = 3 \).

We will reformulate the quantum Chevalley formula stated in [3] using the basis

\[
G := \{ \iota, [hp], \alpha_1, \alpha_2, \ldots, \alpha_{18} \}.
\]

**Proposition 1.** The following equalities hold by [3, Proposition 4.2.1].

\begin{align*}
\hat{c}_1(\iota) &= 5[hp] \\
\hat{c}_1(hp) &= 10\alpha_1 + 5\alpha_2 \\
\hat{c}_1(\alpha_1) &= 5\alpha_3 + 5\alpha_4 \\
\hat{c}_1(\alpha_2) &= 10\alpha_3 + 5\alpha_5 \\
\hat{c}_1(\alpha_3) &= 10\alpha_6 + 5\alpha_7 + 5\alpha_8 \\
\hat{c}_1(\alpha_4) &= 5\alpha_6 + 10\alpha_7 \\
\hat{c}_1(\alpha_5) &= 5\alpha_8 \\
\hat{c}_1(\alpha_6) &= 10\alpha_9 + 5\alpha_{10} + 5\alpha_{11} \\
\hat{c}_1(\alpha_7) &= 5\alpha_{10} \\
\hat{c}_1(\alpha_8) &= 5\alpha_{11} + 5\iota \\
\hat{c}_1(\alpha_9) &= 5\alpha_{12} + 5\alpha_{13} \\
\hat{c}_1(\alpha_{10}) &= 10\alpha_{13} + 5\alpha_{14} \\
\hat{c}_1(\alpha_{11}) &= 5\alpha_{12} + 5\alpha_{14} + 5[hp] \\
\hat{c}_1(\alpha_{12}) &= 5\alpha_{15} + 5\alpha_1 \\
\hat{c}_1(\alpha_{13}) &= 5\alpha_{15} + 5\alpha_{16} \\
\hat{c}_1(\alpha_{14}) &= 5\alpha_{15} + 5\alpha_2 \\
\hat{c}_1(\alpha_{15}) &= 5\alpha_{17} + 5\alpha_3 \\
\hat{c}_1(\alpha_{16}) &= 5\alpha_{17} + 5\alpha_5 \\
\hat{c}_1(\alpha_{17}) &= 5\alpha_{18} + 5\alpha_6 + 5\alpha_8 \\
\hat{c}_1(\alpha_{18}) &= 5\alpha_9 + 5\alpha_{11} + 5\iota
\end{align*}

The following is the quantum Chevalley Bruhat graph of the Fano variety \( X \) in case (1) for \( n = 3 \).

**Figure 3.** Class (1) QCBG: strongly connected, cycle of length \( r = 5 \).
Lemma 6. Property $\mathcal{O}$ holds when $X$ is case (1) with $n = 3$ of Pasquier’s list.

Proof. The coefficients that appear in the equations in Proposition 1 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 3, and the cycle $\alpha_{18}\alpha_{11}\alpha_{14}\alpha_{15}\alpha_{17}\alpha_{18}$ has length $r = 5$. □

This graph is a geometric representation of Property $\mathcal{O}$. The eigenvalues of $\hat{c}_1(\alpha_i) = 5(hp \ast \alpha_i)|_{q=1}$ are plotted on $\mathbb{C}$ and then eigenvalues of equal modulus are connected by lines that form regular polygons.

3.2. Case (2). Again, we reformulate the quantum Chavelley formula from [3] using the basis
$$ G := \{ \iota, [hp], \alpha_1, \alpha_2, \ldots, \alpha_{12} \}.$$

Proposition 2. The following equalities hold by [3] Proposition 4.3.1].

$$
\begin{align*}
\hat{c}_1(\iota) &= 7[hp] \\
\hat{c}_1(hp) &= 7\alpha_1 \\
\hat{c}_1(\alpha_1) &= 14\alpha_2 + 7\alpha_3 \\
\hat{c}_1(\alpha_2) &= 7\alpha_4 + 7\alpha_5 \\
\hat{c}_1(\alpha_3) &= 7\alpha_5 \\
\hat{c}_1(\alpha_4) &= 7\alpha_6 + 7\alpha_7 \\
\hat{c}_1(\alpha_5) &= 7\alpha_7 \\
\hat{c}_1(\alpha_6) &= 7\alpha_8 \\
\hat{c}_1(\alpha_7) &= 7\alpha_8 + 7\alpha_9 \\
\hat{c}_1(\alpha_8) &= 7\alpha_{10} \\
\hat{c}_1(\alpha_9) &= 7\alpha_{10} + 7\iota \\
\hat{c}_1(\alpha_{10}) &= 7\alpha_{11} + 7[hp] \\
\hat{c}_1(\alpha_{11}) &= 7\alpha_{12} + 7\alpha_1 \\
\hat{c}_1(\alpha_{12}) &= 7\alpha_2
\end{align*}
$$
The associated quantum Chevalley Bruhat graph is

![Graph Image]

**Figure 5.** Class (2) QCBG: Strongly connected, cycle of length $r = 7$.

**Lemma 7.** Property $O$ holds when $X$ is case (2) of Pasquier’s list.

**Proof.** The coefficients that appear in the equations in Proposition 2 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 5, and the cycle $\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10}\alpha_{11}\alpha_{12}$ has length $r = 7$. □

This graph is a geometric representation of Property $O$. The eigenvalues of $\hat{c}_1(\alpha_i) = 7(hp \star \alpha_i)|_{q=1}$ are plotted on $\mathbb{C}$ and then eigenvalues of equal modulus are connected by lines that form regular polygons.
3.3. Case(5). Again, we reformulate the quantum Chavelley formula from [3] using the basis

\[ G := \{ \iota, [hp], \alpha_1, \alpha_2, \ldots, \alpha_{10} \}. \]

**Proposition 3.** The following equalities hold by [3, Proposition 4.5.1].

\[
\begin{align*}
\hat{c}_1(\iota) &= 4[hp] \\
\hat{c}_1(hp) &= 12\alpha_1 + 4\alpha_2 \\
\hat{c}_1(\alpha_1) &= 8\alpha_3 + 4\alpha_4 \\
\hat{c}_1(\alpha_2) &= 4\alpha_4 \\
\hat{c}_1(\alpha_3) &= 12\alpha_5 + 4\alpha_6 \\
\hat{c}_1(\alpha_4) &= 4\alpha_6 + 4\iota \\
\hat{c}_1(\alpha_5) &= 4\alpha_7 + 4\alpha_8 \\
\hat{c}_1(\alpha_6) &= 8\alpha_7 + 4[hp] \\
\hat{c}_1(\alpha_7) &= 4\alpha_9 + 4\alpha_1 \\
\hat{c}_1(\alpha_8) &= 4\alpha_9 + 4\alpha_2 \\
\hat{c}_1(\alpha_9) &= 4\alpha_{10} + 4\alpha_3 + 4\alpha_4 \\
\hat{c}_1(\alpha_{10}) &= 4\alpha_5 + 4\alpha_6 + 8\iota
\end{align*}
\]

The associated quantum Chevalley Bruhat graph is

![Quantum Chevalley Bruhat Graph](image)

**Figure 7.** Class (5) QCBG: Strongly connected, cycle of length \( r = 4 \).

**Lemma 8.** Property \( O \) holds when \( X \) is case (5) of Pasquier’s list.

**Proof.** The coefficients that appear in the equations in Proposition 3 are the entries of the matrix representation of \( \hat{c}_1 \). Therefore, the matrix representation of \( \hat{c}_1 \) is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 7, and the cycle \( \alpha_{10}\alpha_6\alpha_7\alpha_9\alpha_{10} \) has length \( r = 4 \). \( \Box \)
This graph is a geometric representation of Property $\mathcal{O}$. The eigenvalues of 
\[ \hat{c}_1(\alpha_i) = 4(hp \star \alpha_i) \mid_{q=1} \]
are plotted on $\mathbb{C}$ and then eigenvalues of equal modulus are connected by lines that form regular polygons.

Theorem 1 follows from Lemmas 6, 7, and 8.

REFERENCES


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