Conjecture $\mathcal{O}$ Holds for Some Horospherical Varieties of Picard Rank 1

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Introduction

Property $\mathcal{O}$ for an arbitrary complex, Fano manifold X is a statement about the eigenvalues of the linear operator obtained from the quantum multiplication of the anticanonical class of X. Pasquier listed the non-homogenous horospherical varieties of Picard rank 1 into five classes. Property $\mathcal{O}$ has already been shown to hold for one class, the odd symplectic Grassmannian. We will show that Property $\mathcal{O}$ holds for two more classes and an example in a third class of Pasquier's list. The theory of Perron-Frobenius reduces our proofs to be graph theoretic.

Background

Consider the projective plane $\mathbb{P}^2$. It resembles $\mathbb{C}^2$ where we allow parallel lines to intersect at infinity. Let pt be a point and hp be a hyperplane (i.e. a line in $\mathbb{P}^2$).

The notation $[pt]$, $[hp]$, and $[\mathbb{P}^2]$ denote a point, a hyperplane, and the projective plane in general position (e.g. two lines are in general position if they intersect at a single point). For notation, we will use the Poincaré duals:

$$[\mathbb{P}^2]^\vee = [pt], [hp]^\vee = [hp], [pt]^\vee = [\mathbb{P}^2].$$

The objects $[pt]$, $[hp]$, and $[\mathbb{P}^2]$ over $\mathbb{Z}[q]$ is a basis for the quantum cohomology ring $\mathrm{QH}^*(\mathbb{P}^2)$. Addition "+" is formal, and multiplication "$\star$" counts geometric objects.

Example 1: $[\mathbb{P}^2] \star [hp] = 1q^0[hp]^\vee$ since there is exactly one point (i.e. a degree 0 curve) that intersects $\mathbb{P}^2$ and two general hyperplanes.

Example 2: $[hp] \star [pt] = 1q^1[pt]^\vee$ since there is exactly one hyperplane (i.e. a degree 1 curve) that intersects a hyperplane and two points in general position.

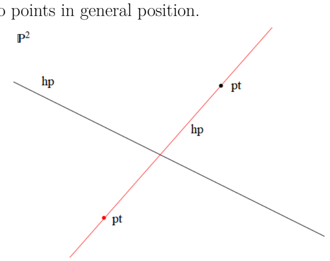


Figure: An example of $[hp] \star [pt] = 1q^1[pt]^\vee$

Multiplication Table of $\mathrm{QH}^*(\mathbb{P}^2)$

$[\mathbb{P}^2] \star [\mathbb{P}^2] = 1q^0[pt]^\vee = [\mathbb{P}^2]$
$[\mathbb{P}^2] \star [hp] = 1q^0[hp]^\vee = [hp]$
$[\mathbb{P}^2] \star [pt] = 1q^0[\mathbb{P}^2]^\vee = [pt]$
$[hp] \star [hp] = 1q^0[\mathbb{P}^2]^\vee = [pt]$
$[hp] \star [pt] = 1q^1[pt]^\vee = 1q^1[\mathbb{P}^2]$
$[pt] \star [pt] = 1q^1[hp]^\vee = 1q^1[hp]$

Background (cont.)

Consider the linear operator $\hat{c}_1$ obtained from the multiplication of the anticanonical class $3[hp]$ and setting $q = 1$.

$3[hp]\star$	$[pt]$	$[hp]$	$[\mathbb{P}^2]$
$[pt]$	0	0	3
$[hp]$	3	0	0
$[\mathbb{P}^2]$	0	3	0

The characteristic polynomial is $\lambda^3 - 27$. The eigenvalues of $\hat{c}_1$ are three times the third roots of unity, $3, 3e^{\frac{2\pi}{3}i}, 3e^{\frac{4\pi}{3}i}$. The Fano index of $\mathbb{P}^2$ is 3. This is an example where Conjecture $\mathcal{O}$ holds.

Conjecture $\mathcal{O}$

Let X be a Fano manifold. The equations arising from the multiplication of hyperplane classes is the quantum Chevalley formula. Let $\hat{c}_1$ denote the linear operator arising from the multiplication of the anticanonical class when $q = 1$. The following is Conjecture $\mathcal{O}$ by Galkin, Golyshev, and Iritani.

We denote

$$\delta_0 := \max\{|\delta| \ : \ \delta \text{ is an eigenvalue of } \hat{c}_1\}$$

The following conditions hold.

- The real number δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity one.
- If δ is an eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0\zeta$ for some r-th root of unity $\zeta \in \mathbb{C}$, where r is the Fano index of X.

Horospherical Varieties

Let G be a complex reductive group. A G-variety is a reduced scheme of finite type over the field of complex numbers $\mathbb{C}$, equipped with an algebraic action of G. Let B be a Borel subgroup of G. A G-variety X is called spherical if X has a dense B-orbit. Let X be a G-spherical variety and let H be the stabilizer of a point in the dense G-orbit in X. The variety X is called **horospherical** if H contains a conjugate of the maximal unipotent subgroup of G contained in the Borel subgroup B.

Horospherical varieties of Picard rank 1 are either **homogeneous** or can be constructed in a uniform way via a triple $(\mathrm{Type}(G), \omega_Y, \omega_Z)$ of representation-theoretic data, where $\mathrm{Type}(G)$ is the semisimple Lie type of the reductive group G and ω_Y, ω_Z are the fundamental weights. Pasquier classified the possible triples in five classes:

- $(B_n, \omega_{n-1}, \omega_n)$ with $n \geq 3$;
- $(B_3, \omega_1, \omega_3)$;
- $(C_n, \omega_m, \omega_{m-1})$ with $n \geq 2$ and $m \in [2, n]$;
- $(F_4, \omega_2, \omega_3)$;
- $(G_2, \omega_1, \omega_2)$.

Theorem

If F belongs to the classes (1) for $n = 3$, (2), (3), and (5) of Pasquier's list, then Conjecture $\mathcal{O}$ holds for F.

- The homogenous case was proved by Cheong and Li (2017).
- Class (3) was proved by Li, Mihalcea, and Shifler (2017).
- We prove the theorem for the classes (1) for $n = 3$, (2), and (5).
- Explicit calculations of Chevalley formulas is a key ingredient to our proofs. (Gonzales, Pech, Perrin, and Samokhin (2018))

The **quantum Chevalley Bruhat graph** of a quantum cohomology ring is a directed graph where

- the vertices are basis elements of the quantum cohomology ring;
- there is a directed edge $\alpha \to \beta$ if β appears with a positive coefficient in the expansion of $\hat{c}_1(\alpha)$.

The techniques involving Perron-Frobenius theory used by Cheong, Li, Mihalcea, and Shifler imply the following lemma:

Lemma

If the following conditions hold for a Fano variety F:

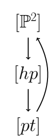
- the matrix representation of $\hat{c}_1$ is nonnegative;
- the quantum Chevalley Bruhat graph of F is strongly connected;
- there exists a cycle of length r, the Fano index, in the quantum Chevalley Bruhat graph of F;

then Conjecture $\mathcal{O}$ holds for F.

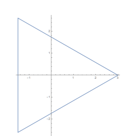
Looking back at the example of $\mathbb{P}^2$, we see that the conditions of the lemma are all met, and therefore Conjecture $\mathcal{O}$ holds.

$$\begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Nonnegative



(b) Strongly connected, $r = 3$



(c) Eigenvalues plotted on $\mathbb{C}$

The lemma is generalized in a recent paper by Hu, Ke, Li, and Yang to include some matrices with negative entries.

Proof for Class (1), $n = 3$

We will show the complete argument for the Class (1), $n = 3$ since the proofs for Class (2) and Class (5) are similar. The Fano index for Class (1), $n = 3$ is $r = 5$, the anticanonical class is $5[hp]$, and the quantum cohomology ring over $\mathbb{Z}[q]$ has the basis

$$G := \{\iota, [hp], \alpha_1, \alpha_2, \cdots, \alpha_{18}\}.$$

Proof for Class (1), $n = 3$ (cont.)

The linear operator $\hat{c}_1$ is given by

$$\hat{c}_1(\alpha) = (5[hp] \star \alpha)|_{q=1} \text{ for any } \alpha \in G.$$

$\hat{c}_1(\iota) = 5[hp]$
$\hat{c}_1(hp) = 10\alpha_1 + 5\alpha_2$
$\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4$
$\hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5$
$\hat{c}_1(\alpha_3) = 10\alpha_6 + 5\alpha_7 + 5\alpha_8$
$\hat{c}_1(\alpha_4) = 5\alpha_6 + 10\alpha_7$
$\hat{c}_1(\alpha_5) = 5\alpha_8$
$\hat{c}_1(\alpha_6) = 10\alpha_9 + 5\alpha_{10} + 5\alpha_{11}$
$\hat{c}_1(\alpha_7) = 5\alpha_{10}$
$\hat{c}_1(\alpha_8) = 5\alpha_{11} + 5\iota$
$\hat{c}_1(\alpha_9) = 5\alpha_{12} + 5\alpha_{13}$
$\hat{c}_1(\alpha_{10}) = 10\alpha_{13} + 5\alpha_{14}$
$\hat{c}_1(\alpha_{11}) = 5\alpha_{12} + 5\alpha_{14} + 5[hp]$
$\hat{c}_1(\alpha_{12}) = 5\alpha_{15} + 5\alpha_1$
$\hat{c}_1(\alpha_{13}) = 5\alpha_{15} + 5\alpha_{16}$
$\hat{c}_1(\alpha_{14}) = 5\alpha_{15} + 5\alpha_2$
$\hat{c}_1(\alpha_{15}) = 5\alpha_{17} + 5\alpha_3$
$\hat{c}_1(\alpha_{16}) = 5\alpha_{17} + 5\alpha_5$
$\hat{c}_1(\alpha_{17}) = 5\alpha_{18} + 5\alpha_6 + 5\alpha_8$
$\hat{c}_1(\alpha_{18}) = 5\alpha_9 + 5\alpha_{11} + 10\iota$

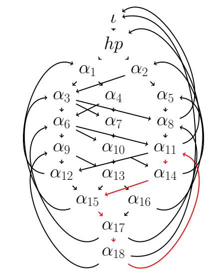


Figure: Strongly connected, cycle of length $r = 5$.

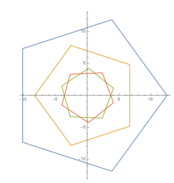


Figure: Class (1), $n = 3$ eigenvalues plotted on $\mathbb{C}$

References

[1] D. Cheong, C. Li, *On the Conjecture $\mathcal{O}$ of GGI for G/P.* Advances in Mathematics, 306 (2017), 704-721.

[2] S. Galkin, V. Golyshev, and H. Iritani, *Gamma Classes and Quantum Cohomology of Fano Manifolds: Gamma Conjectures.* Duke Mathematical Journal, 165 (2016) no. 11, 2005-2077.

[3] R. Gonzales, C. Pech, N. Perrin and A. Samokhin, *Geometry of Horospherical Varieties of Picard Rank One,* (2018), arXiv:1803.05063.

[4] C. Li, L. Mihalcea, and R. Shifler, *Conjecture $\mathcal{O}$ Holds for the Odd Symplectic Grassmannian.* (2017), arXiv:1706.00744.

[5] H. Minc. *Nonnegative matrices.* (1988), Wiley.

[6] B. Pasquier, *On Some Smooth Projective Two-orbit Varieties with Picard Number 1.* Mathematische Annalen, 344 (2009) no. 4, 963-987.

[7] B. Pasquier, *Variétés horosphèriques de Fano.* Available at `http://tel.archives-ouvertes.fr/docs/00/11/60/77/PDF/Pasquier2006/pdf`.

[8] J. Hu, H. Ke, C. Li, and T. Yang *Gamma Conjecture I for Del Pezzo Surfaces.*(2019) Available at arXiv:1901.01748.