POSITIVITY DETERMINES THE QUANTUM COHOMOLOGY OF THE ODD SYMPLECTIC GRASSMANNIAN OF LINES

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Abstract. Let $\mathrm{IG} := \mathrm{IG}(2, 2n+1)$ denote the odd symplectic Grassmannian of lines which is a horospherical variety of Picard rank 1. The quantum cohomology ring $\mathrm{QH}^*(\mathrm{IG})$ has negative structure constants. For $n \geqslant 3$, we give a positivity condition that implies the quantum cohomology ring $\mathrm{QH}^*(\mathrm{IG})$ is the only quantum deformation of the cohomology ring $\mathrm{H}^*(\mathrm{IG})$ up to the scaling of the quantum parameter. This is a modification of a conjecture by Fulton.

1. Introduction

Let $\mathrm{IG} := \mathrm{IG}(2, 2n+1)$ denote the odd symplectic Grassmannian of lines which is a horospherical variety of Picard rank 1. This is the parameterization of two dimensional subspaces of $\mathbb{C}^{2n+1}$ that are isotropic with respect to a general skew-symmetric form. The quantum cohomology ring $(\mathrm{QH}^*(\mathrm{IG}), \star)$ is a graded algebra over $\mathbb{Z}[q]$ where q is the quantum parameter and $\deg q = 2n$. The ring has a Schubert basis given by $\{\tau_\lambda : \lambda \in \Lambda\}$ where

$$\Lambda := \{(\lambda_1, \lambda_2) : 2n-1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant -1,\ \lambda_1 > n-2 \Rightarrow \lambda_1 > \lambda_2, \text{ and } \lambda_2 = -1 \Rightarrow \lambda_1 = 2n-1\}.$$

We will often write τ_i in place of $\tau_{(i,0)}$. We define $|\lambda| = \lambda_1 + \lambda_2$ for any $\lambda \in \Lambda$. Then $\deg(\tau_\lambda) = |\lambda|$. The ring multiplication is given by $\tau_\lambda \star \tau_\mu = \sum_{\nu,d} c_{\lambda,\mu}^{\nu,d} q^d \tau_\nu$ where $c_{\lambda,\mu}^{\nu,d}$ is the degree d Gromov-Witten invariant of τ_λ, τ_μ, and the Poicaré dual of τ_ν. Unlike the homogeneous G/P case, the Gromov-Witten invariants may be negative. For example, in $\mathrm{IG}(2, 5)$, we have

$$\tau_{(3,-1)} \star \tau_{(3,-1)} = \tau_{(3,1)} - q \text{ and } \tau_{(2,1)} \star \tau_{(3,-1)} = -\tau_{(3,2)} + q\tau_1.$$

The quantum Pieri rule has only non-negative coefficients and is stated in Proposition 2.2. See [Pec13, MS19, GPPS19] for more details on IG.

Definition 1.1. For any given collection of constants $\{a_\mu \in \mathbb{Q} : \mu \in \Lambda\}$, a quantum deformation with the corresponding basis $\{\sigma_\lambda : \lambda \in \Lambda\}$ is defined as a solution to the following system:

$$\tau_\lambda = \sigma_\lambda + \sum_{j \geqslant 1} \left(\sum_{|\mu| + 2nj = |\lambda|} a_\mu q^j \sigma_\mu \right), \lambda \in \Lambda.$$

Remark 1.2. It is always possible to re-scale the quantum parameter q by a positive factor $\alpha > 0$, or equivalently, multiply each Gromov-Witten invariant $c_{\lambda,\mu}^{\nu,d}$ by α^{-d}. We only consider the $\alpha = 1$ case in this manuscript.

To contextualize the significance of quantum deformations we review the following conjecture by Fulton for Grassmannians and its extension to a more general case by Buch and Wang in [BW21, Conjecture 1].

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Conjecture 1. *Let $X = G/P$ be any flag variety of simply laced Lie type. Then the Schubert basis of $\mathrm{QH}^*(X)$ is the only homogeneous $\mathbb{Q}[q]$-basis that deforms the Schubert basis of $H^*(X, \mathbb{Q})$ and multiplies with non-negative structure constants.*

This conjecture is shown to hold for any Grassmannian and a few other examples in [BW21]. Li and Li proved the result for symplectic Grassmannians $\mathrm{IG}(2, 2n)$ with $n \geqslant 3$ in [LL23]. The condition that the root system of G be simply laced is necessary since the conjecture fails to hold for the Lagrangian Grassmannian $\mathrm{IG}(2, 4)$ as shown in [BW21, Example 6]. However, this conjecture is not applicable to $\mathrm{IG}(2, 2n + 1)$ since negative coefficients appear in quantum products for any n. We are able to modify the positivity condition on Fulton's conjecture to arrive at a uniqueness result for quantum deformations.

Definition 1.3. For $\mathrm{IG}(2, 2n + 1)$ we will use (**) to denote the condition that the coefficients of the quantum multiplication of $\sigma_{(1,1)}$ and any σ_μ in the basis $\{\sigma_\lambda : \lambda \in \Lambda\}$ are polynomials in q with non-negative coefficients.

We are ready to state the main result.

Theorem 1.4. *Let $n \geqslant 3$. Suppose that $\{\sigma_\lambda : \lambda \in \Lambda\}$ is a quantum deformation of the Schubert basis $\{\tau_\lambda : \lambda \in \Lambda\}$ of $\mathrm{QH}^*(\mathrm{IG})$ such that Condition (**) holds. Then $\tau_\lambda = \sigma_\lambda$ for all $\lambda \in \Lambda$.*

Remark 1.5. The methods used in this manuscript are motivated by those of Li and Li in [LL23]. In particular, multiplication by $\tau_{(1,1)}$, which is not a generator, is all that we use to establish the uniqueness of the quantum deformation.

In Section 2 we prove the main result for the $|\lambda| < 2n$ case, state the quantum Pieri rule, and give identities for later in the paper; in Section 3 we prove the main result for the $|\lambda| = 2n$ case; and in Section 4 we prove the main result for the $|\lambda| > 2n$ case. Theorem 1.4 follows from Propositions 2.1, 3.1, and 4.2.

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2. Preliminaries

We begin the section with a proposition that reduces the number of possible quantum deformations that we need to check. This is accomplished by using the grading. The proposition also states our main result for the $|\lambda| < 2n$ case.

Proposition 2.1. *We have the following results.*

1. *We have that $\tau_\lambda = \sigma_\lambda + \sum_{|\mu|+2n=|\lambda|} a_\mu q \sigma_\mu$.*
2. *If $|\lambda| < 2n$ then $\tau_\lambda = \sigma_\lambda$.*

Proof. The first part follows since $|\lambda| \leqslant \dim(\mathrm{IG}(2, 2n + 1) = 4n - 3 < 4n = 2 \deg q$ for any $\lambda \in \Lambda$. The second part follows immediately from the grading. □

Next we state the quantum Pieri rule for $\mathrm{IG}(2, 2n + 1)$.

Proposition 2.2. [Pec13, Theorem 1] *The quantum Pieri rule.*

$$\tau_1 \star \tau_{a,b} = \begin{cases} \tau_{a+1,b} + \tau_{a,b+1} & \textit{if } a + b \neq 2n - 3 \textit{ and } a \neq 2n - 1, \\ \tau_{a,b+1} + 2\tau_{a+1,b} + \tau_{a+2,b-1} & \textit{if } a + b = 2n - 3, \\ \tau_{2n-1,b+1} + q\tau_b & \textit{if } a = 2n - 1 \textit{ and } 0 \leqslant b \leqslant 2n - 3, \\ \tau_{2n-1} & a = 2n - 1 \textit{ and } b = -1, \\ q(\tau_{2n-1,-1} + \tau_{2n-2}) & a = 2n - 1 \textit{ and } b = 2n - 2. \end{cases}$$

$$\tau_{1,1} \star \tau_{a,b} = \begin{cases} \tau_{a+1,b+1} & \text{if } a+b \neq 2n-4, 2n-3 \text{ and } a \neq 2n-1, \\ \tau_{a+2,b} + \tau_{a+1,b+1} & \text{if } a+b = 2n-4 \text{ or } 2n-3, \\ q\tau_{b+1} & \text{if } a = 2n-1 \text{ and } b \neq 2n-3, \\ q(\tau_{2n-1,-1} + \tau_{2n-2}) & a = 2n-1 \text{ and } b = 2n-3. \end{cases}$$

Lemma 2.3. *We have the following identities.*

(1) *Let $t \leqslant n-2$. Then*

$$\sigma_{(t,t)} = \tau_{(t,t)} = \Pi_{i=1}^{t}\tau_{(1,1)} = \Pi_{i=1}^{t}\sigma_{(1,1)}.$$

(2) *Let $|\lambda| \geqslant 2n$ and $t := 2n - \lambda_1$.*

(a) *If $\lambda_2 + t \neq 2n-2$. Then*

$$\left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\lambda = q\tau_{(\lambda_2+t)}.$$

(b) *If $\lambda_2 + t = 2n-2$. Then*

$$\left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\lambda = q\tau_{(2n-1,-1)} + q\tau_{(2n-2)}.$$

(3) *We have that*

$$\Pi_{i=1}^{n-1}\tau_{(1,1)} = \tau_{(n,n-2)}.$$

(4) *If $2t + |\mu| \leqslant 2n-3$ and $t \leqslant n-2$ then*

$$\left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\mu = \tau_{(\mu_1+t,\mu_2+t)}.$$

(5) *If $2t + |\mu| = 2n-2$ or $2n-1$ and $t \leqslant n-2$ then*

$$\left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\mu = +\tau_{(\mu_1+t+1,\mu_2+t-1)} + \tau_{(\mu_1+t,\mu_2+t)}.$$

Proof. Part (1) is clear since $2t \leqslant 2n-4$. For Part (2), $\tau_{(1,1)} \star \tau_{(t-1,t-1)} \star \tau_\lambda = \tau_{(1,1)} \star \tau_{(2n-1,\lambda_2+t-1)} = q\tau_{(\lambda_2+t)}$ or $\tau_{(1,1)} \star \tau_{(t-1,t-1)} \star \tau_\lambda = \tau_{(1,1)} \star \tau_{(2n-1,\lambda_2+t-1)} = q\tau_{(2n-1,-1)} + q\tau_{(2n-2)}$. For Part (3), $\tau_{(1,1)} \star \Pi_{i=1}^{n-2}\tau_{(1,1)} = \tau_{(1,1)} \star \tau_{(n-2,n-2)} = \tau_{(n,n-2)}$. Part (4) is clear. For Part (5), we have $\tau_{(1,1)} \star \left(\Pi_{i=1}^{t-1}\tau_{(1,1)}\right) \star \tau_\mu = \tau_{(1,1)} \star \tau_{(\mu_1+t-1,\mu_2+t-1)} = \tau_{(\mu_1+t,\mu_2+t)} + \tau_{(\mu_1+t+1,\mu_2+t-1)}$. This completes the proof. □

3. The $|\lambda| = 2n$ case

In this section we will assume that $|\lambda| = 2n$. The main proposition of this section is stated next.

Proposition 3.1. *Let $|\lambda| = 2n$. If $\tau_\lambda = \sigma_\lambda + aq$ and Condition (**) holds then $\tau_\lambda = \sigma_\lambda$.*

Proof. By Proposition 2.1 it must be the case that $\tau_\lambda = \sigma_\lambda + aq$. We show $a \leqslant 0$ in two parts. Lemma 3.2 considers the $\lambda_1 \geqslant n+2$ case and Lemma 3.3 considers the $\lambda = (n+1, n-1)$ case. We show $a \geqslant 0$ in Lemma 3.4 as a straightforward application of the quantum Pieri rule. This completes the proof. □

Lemma 3.2. *Let $|\lambda| = 2n$ and $\lambda_1 \geqslant n+2$. If $\tau_\lambda = \sigma_\lambda + aq$ and Condition (**) holds then $a \leqslant 0$.*

Proof. Let $t := 2n - \lambda_1 \leqslant n-2$. Note that $t + \lambda_1 = 2n$. Then we have the following by multiplying $\sigma_\lambda = \tau_\lambda - aq$ by $\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right)$ and using Part (1) of Lemma 2.3.

$$\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_\lambda = \tau_{(t,t)} \star \tau_\lambda - a\sigma_{(t,t)}q.$$

By Part (2) of Lemma 2.3 we have $\tau_{(t,t)} \star \tau_\lambda = q\tau_{(\lambda_2+t)} = q\sigma_{(\lambda_2+t)}$. So,

$$\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_\lambda = q\sigma_{(\lambda_2+t)} - a\sigma_{(t,t)}q.$$

It follows from Condition (**) that $a \leqslant 0$. □

We will now prove $a \leqslant 0$ for the $\lambda = (n+1, n-1)$ case.

Lemma 3.3. *Let $\lambda = (n+1, n-1)$. If $\tau_\lambda = \sigma_\lambda + aq$ and Condition (**) holds then $a \leqslant 0$.*

Proof. Recall from Part (3) of Lemma 2.3 that $\Pi_{i=1}^{n-1}\tau_{(1,1)} = \tau_{(n,n-2)}$ and from Part (2) of Lemma 2.3 we have that $\left(\Pi_{i=1}^{n-1}\tau_{(1,1)}\right) \star \tau_\lambda = q\tau_{(2n-1,-1)} + q\tau_{(2n-2)}$. Multiplying $\sigma_\lambda = \tau_\lambda - aq$ by $\left(\Pi_{i=1}^{n-1}\sigma_{(1,1)}\right)$ and substituting in the identities yields

$$\begin{aligned}\left(\Pi_{i=1}^{n-1}\sigma_{(1,1)}\right) \star \sigma_\lambda &= \left(\Pi_{i=1}^{n-1}\tau_{(1,1)}\right) \star \tau_\lambda - a\left(\Pi_{i=1}^{n-1}\tau_{(1,1)}\right) q \\ &= q\tau_{(2n-1,-1)} + q\tau_{(2n-2)} - aq\tau_{(n,n-2)} \\ &= q\sigma_{(2n-1,-1)} + q\sigma_{(2n-2)} - aq\sigma_{(n,n-2)}.\end{aligned}$$

It follows from Condition (**) that $a \leqslant 0$. $\square$

We conclude the section by showing that $a \geqslant 0$ in the next lemma.

Lemma 3.4. *Let $|\lambda| = 2n$. If $\tau_\lambda = \sigma_\lambda + aq$ and Condition (**) holds then $a \geqslant 0$.*

Proof. Let $\lambda^j = (n+1+j, n-1-j)$ for all $j = 0, 1, 2, ..., n-2$. Assume that $\tau_{\lambda^j} = \sigma_{\lambda^j} + a_j q$. Then for all $0 \leqslant j \leqslant n-2$ it follows from the quantum Pieri rule that $\tau_{(1,1)} \star \tau_{(n+j,n-2-j)} = \tau_{\lambda^j}$. Since $\tau_{(n+j,n-2-j)} = \sigma_{(n+j,n-2-j)}$ by Part (2) of Lemma 2.1, we have that

$$\sigma_{(1,1)} \star \sigma_{(n+j,n-2-j)} = \tau_{(1,1)} \star \tau_{(n+j,n-2-j)} = \tau_{\lambda^j} = \sigma_{\lambda^j} + a_j q.$$

It follows from Condition (**) that $a_j \geqslant 0$ for all $j = 0, \cdots, n-2$. $\square$

4. The $|\lambda| > 2n$ case

In this section we will assume that $|\lambda| > 2n$. Recall that by Proposition 2.1 it must be the case that $\tau_\lambda = \sigma_\lambda + \sum_{|\mu|+2n=|\lambda|} a_\mu q \sigma_\mu$.

Lemma 4.1. *Let $|\lambda| > 2n$. If $\tau_\lambda = \sigma_\lambda + \sum_{|\mu|+2n=|\lambda|} a_\mu q \sigma_\mu$ and Condition (**) holds then $a_\mu \leqslant 0$ or there is a μ' such that $a_\mu + a_{\mu'} \leqslant 0$.*

Proof. If $|\lambda| > 2n$ then $\lambda_1 \geqslant n+1$. Let $t := 2n - \lambda_1 \leqslant n-1$. Let $A(\lambda) = \sigma_{\lambda_2+t}$ if $\lambda_2 + t \neq 2n-2$ and $A(\lambda) = \sigma_{(2n-1,-1)} + \sigma_{(2n-2)}$ if $\lambda_2 + t = 2n-2$. We will multiply $\sigma_\lambda = \tau_\lambda - \sum_{|\mu|+2n=|\lambda|} a_\mu q \sigma_\mu$ by $\left(\Pi_{i=1}^{t}\tau_{(1,1)}\right)$. By Part (2) of Lemma 2.3 we have that $\left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\lambda = qA(\lambda)$. Since $\lambda_2 + t < \lambda_1 + t = 2n$, we have that

$$\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_\lambda = qA(\lambda) - \left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \left(\sum_{|\mu|+2n=|\lambda|} a_\mu q \sigma_\mu\right).$$

Next observe that $2t + |\mu| = 2t + |\lambda| - 2n = 2n - \lambda_1 + \lambda_2 \leqslant 2n-1$. So, one of the following must occur:

- If $2t + |\mu| \leqslant 2n-3$ then by Part (4) of Lemma 2.3 we have
$$\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_\mu = \left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\mu = \tau_{(\mu_1+t,\mu_2+t)} = \sigma_{(\mu_1+t,\mu_2+t)}.$$
- If $2t + |\mu| = 2n-1$ or $2n-2$ then by Part (5) of Lemma 2.3 we have
$$\begin{aligned}\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_\mu &= \left(\Pi_{i=1}^{t}\tau_{(1,1)}\right) \star \tau_\mu = \tau_{(\mu_1+t,\mu_2+t)} + \tau_{(\mu_1+t+1,\mu_2+t-1)} \\ &= \sigma_{(\mu_1+t+1,\mu_2+t-1)} + \sigma_{(\mu_1+t,\mu_2+t)}.\end{aligned}$$

Then $P := \left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_\lambda$ equals the following where terms are omitted when they do not satisfy the ring grading.

$$\begin{aligned}
P &= qA(\lambda) - \left(\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|\leqslant 2n-3}} a_\mu q \sigma_{(\mu_1+t,\mu_2+t)} \right) - \left(\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|=2n-1}} a_\mu q \left(\sigma_{(\mu_1+t+1,\mu_2+t-1)} + \sigma_{(\mu_1+t,\mu_2+t)}\right) \right) \\
&- \left(\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|=2n-2}} a_\mu q \left(\sigma_{(\mu_1+t+1,\mu_2+t-1)} + \sigma_{(\mu_1+t,\mu_2+t)}\right) \right).
\end{aligned}$$

We have the following two equalities that will be used to precisely write the summations for the $2t+|\mu| = 2n-1$ and $2t+|\mu| = 2n-2$ cases.

$$\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_{(2n-1-2t-i,i)} = \sigma_{(2n-t-i,t-1+i)} + \sigma_{(2n-1-t-i,t+i)} \text{ for } 0 \leqslant i < n-1-t.$$

$$\left(\Pi_{i=1}^{t}\sigma_{(1,1)}\right) \star \sigma_{(2n-2-2t-i,i)} = \begin{cases} \sigma_{(2n-1-t-i,t-1+i)} + \sigma_{(2n-2-t-i,t+i)} & : 0 \leqslant i < n-1-t \\ \sigma_{(n,n-2)} & : i = n-1-t. \end{cases}$$

To simplify notation we will let $a_i = a_{(2n-1-2t-i,i)}$ and $b_i = a_{(2n-2-2t-i,i)}$ for $0 \leqslant i \leqslant n-1-t$. Then we have the following identities.

$$\begin{aligned}
\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|=2n-1}} a_\mu q \left(\sigma_{(\mu_1+t+1,\mu_2+t-1)} + \sigma_{(\mu_1+t,\mu_2+t)}\right) &= \sum_{i=0}^{n-1-t} a_i q \left(\sigma_{(2n-t-i,t-1+i)} + \sigma_{(2n-1-t-i,t+i)}\right). \\
\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|=2n-2}} a_\mu q \left(\sigma_{(\mu_1+t+1,\mu_2+t-1)} + \sigma_{(\mu_1+t,\mu_2+t)}\right) &= \left(\sum_{i=0}^{n-2-t} b_i q \left(\sigma_{(2n-1-t-i,t-1+i)} + \sigma_{(2n-2-t-i,t+i)}\right) \right) \\
&+ b_{n-1-t}\sigma_{(n,n-2)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
P &= qA(\lambda) - \left(\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|\leqslant 2n-3}} a_\mu q \sigma_{(\mu_1+t,\mu_2+t)} \right) - \left(\sum_{i=0}^{n-1-t} a_i q \left(\sigma_{(2n-t-i,t-1+i)} + \sigma_{(2n-1-t-i,t+i)}\right) \right) \\
&- \left(\sum_{i=0}^{n-2-t} b_i q \left(\sigma_{(2n-1-t-i,t-1+i)} + \sigma_{(2n-2-t-i,t+i)}\right) \right) - b_{n-1-t}\sigma_{(n,n-2)}.
\end{aligned}$$

Reorganizing the second two sums yields the following equation.

$$\begin{aligned} P &= qA(\lambda) - \left(\sum_{\substack{|\mu|+2n=|\lambda| \\ 2t+|\mu|\leqslant 2n-3}} a_\mu q\sigma_{(\mu_1+t,\mu_2+t)} \right) \\ &- a_0 q\sigma_{(2n-t,t-1)} - \left(\sum_{i=0}^{n-2-t} (a_i + a_{i+1}) q\sigma_{(2n-1-t-i,t+i)} \right) - a_{n-1-t} q\sigma_{(n+1,n-2)} \\ &- b_0 q\sigma_{(2n-1-t,t-1)} - \left(\sum_{i=0}^{n-2-t} (b_i + b_{i+1}) q\sigma_{(2n-2-t-i,t+i)} \right). \end{aligned}$$

When $|\mu| + 2n = |\lambda|$ and $2t + |\mu| \leqslant 2n - 3$, we have that $a_\mu \leqslant 0$ by Condition (**). In the remaining cases, notice by Condition (**) that $a_i + a_{i+1} \leqslant 0$ or $b_i + b_{i+1} \leqslant 0$ for all $0 \leqslant i \leqslant n - 2 - t$. The result follows. □

Proposition 4.2. *Let* $|\lambda| > 2n$. *If* $\tau_\lambda = \sigma_\lambda + \sum_{|\mu|+2n=|\lambda|} a_\mu q\sigma_\mu$ *and Condition (**) holds then* $a_\mu = 0$.

Proof. We proceed by induction. Suppose $\tau_\lambda = \sigma_\lambda$ for all $|\lambda| \leqslant s$ where $s \geqslant 2n$. Consider $|\lambda| = s + 1$. Since $|\lambda| \geqslant 2n + 1$ for $|\lambda| = s + 1$, and by an application of the quantum Pieri rule, we have that $\tau_{(1,1)} \star \tau_{(\lambda_1-1,\lambda_2-1)} = \tau_\lambda$. Observe that $\tau_{(\lambda_1-1,\lambda_2-1)} = \sigma_{(\lambda_1-1,\lambda_2-1)}$ by the inductive hypothesis. Then

$$\sigma_{(1,1)} \star \sigma_{(\lambda_1-1,\lambda_2-1)} = \tau_{(1,1)} \star \tau_{(\lambda_1-1,\lambda_2-1)} = \tau_\lambda = \sigma_\lambda + \sum_{|\mu|+2n=|\lambda|} a_\mu q\sigma_\mu.$$

So, $a_\mu \geqslant 0$ by Condition (**). By Lemma 4.1, either $a_\mu \leqslant 0$ or there is a μ' such that $a_\mu + a_{\mu'} \leqslant 0$. In either case, this implies $a_\mu = 0$ since $a_\mu \geqslant 0$ and $a_{\mu'} \geqslant 0$. The result follows. □

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