CURVE NEIGHBORHOODS AND COMBINATORIAL PROPERTY $\mathcal{O}$ FOR A FAMILY OF ODD SYMPLECTIC PARTIAL FLAG MANIFOLDS

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Abstract. Let E be an odd dimensional complex vector space and $\mathrm{IF} := \mathrm{IF}(1,2;E)$ be the family of odd symplectic partial flag manifold. In this paper we give a full description of the irreducible components of the degree d curve neighborhood of any Schubert variety of IF, study their lattice structure, and prove a combinatorial version of Conjecture $\mathcal{O}$.

1. Introduction

1.1. **Overview.** The degree d curve neighborhood of a subvariety $V \subset X$, denoted $\Gamma_d(V)$, is the closure of the union of all degree d rational curves through V. Curve neighborhoods were introduced in [BCMP13] to prove finiteness of quantum K-theory for X a cominuscule homogeneous space. Let E be an odd dimensional complex vector space and $\mathrm{IF} := \mathrm{IF}(1,2;E)$ be the family of odd symplectic partial flag manifold made up of sequences of vector spaces $(V_1 \subset V_2 \subset E)$ where $\dim V_i = i$ and V_i is isotropic with respect to a (necessarily degenerate) symmetric form. In this paper we give a full description of the irreducible components of the degree d curve neighborhood of any Schubert variety of IF, study their lattice structure, and prove a combinatorial version of Conjecture $\mathcal{O}$.

1.2. **Broader context.** Curve neighborhoods in homogeneous G/P case are irreducible and there are combinatorial models to perform calculations (see [BCMP13, BM15, SW20, Shi]). In [Asl], Aslan shows that the irreducible components of curve neighborhoods in the Affine Flag in Type A have equal dimension. It is shown in [PS22] that curve neighborhoods in the odd symplectic Grassmannian may not be irreducible. In Section 6, we see, for the first time, that reducible curves are an integral part of understanding the geometry of the (combinatorial) quantum Bruhat graph for IF. See Remark 6.4.

1.3. **Curve neighborhoods.** We now discuss curve neighborhoods. Let X be a Fano variety (one could consider X to be smooth). Let $d \in H_2(X,\mathbb{Z})$ be an effective degree. Recall that the moduli space of genus 0, degree d stable maps with two marked points $\overline{\mathcal{M}}_{0,2}(X,d)$ is endowed with two evaluation maps $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{0,2}(X,d) \to X$, $i = 1,2$ which evaluate stable maps at the i-th marked point. Let $\Omega \subset X$ be a closed subvariety. The *curve neighborhood* of Ω is the subscheme

$$\Gamma_d(\Omega) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}\,\Omega) \subset X$$

endowed with the reduced scheme structure.

This notion was introduced by Buch, Chaput, Mihalcea and Perrin [BCMP13] to help study the quantum K-theory ring of cominuscule Grassmannians. It was analyzed further in [BM15, PS22]. Often, estimates for the dimension of the curve neighborhoods provide vanishing conditions for certain Gromov-Witten invariants.

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1.4. **Lattices.** We will study curve neighborhoods through the lens of lattices. That is, let X be a smooth variety containing the subvariety Ω. It's interesting to ask if the set $\{\Gamma_d(\Omega))\}_d$ forms a (distributive) lattice where $\leqslant$ is defined by inclusion of varieties. We will show that this set forms a distributive lattice when $X = \mathrm{IF}$ and Ω is a Schubert variety.

1.5. **Conjecture $\mathcal{O}$.** We will motivate and define the Combinatorial Property $\mathcal{O}$ which is known to correspond with Conjecture $\mathcal{O}$ in the homogeneous G/P and odd symplectic Grassmannian cases. We begin by reviewing Conjecture $\mathcal{O}$ and conclude with its relation to graph theory in Lemma 1.1. In Subsection 1.7 we state the combinatorial versions of the quantum Bruhat graph and the Conjecture $\mathcal{O}$. In Subsection 6.1 we state and prove that Combinatorial Property $\mathcal{O}$ holds for IF in Theorem 6.5.

We recall the precise statement of Conjecture $\mathcal{O}$. Let X be a Fano variety, let $K := K_X$ be the canonical line bundle of X, let X_D be a fundamental divisor of X, and let $c_1(X) := c_1(-K) \in H^2(X)$ be the anticanonical class. The Fano index of X is r, where r is the greatest integer such that $K_X \cong -rX_D$. The small quantum cohomology ring $(QH^*(X), \star)$ is a graded algebra over $\mathbb{Z}[q]$, where q is the quantum parameter. Consider the specialization $H^\bullet(X) := QH^*(X)|_{q=1}$ at $q = 1$. The quantum multiplication by the first Chern class $c_1(X)$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $H^\bullet(X)$:

$$y \in H^\bullet(X) \mapsto \hat{c}_1(y) := (c_1(X) \star y)|_{q=1}.$$

Denote by $\delta_0 := \max\{|\delta| : \delta \text{ is an eigenvalue of } \hat{c}_1\}$. Then Property $\mathcal{O}$ states the following:

(1) The real number δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity one.
(2) If δ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0\gamma$ for some r-th root of unity $\gamma \in \mathbb{C}$, where r is the Fano index of X.

The property $\mathcal{O}$ was conjectured to hold for any Fano, complex manifold X in [GGI16]. If a Fano, complex, manifold has Property $\mathcal{O}$ then we say that the space satisfies Conjecture $\mathcal{O}$. Conjecture $\mathcal{O}$ underlies Gamma Conjectures I and II of Galkin, Golyshev, and Iritani. The Gamma Conjectures refine earlier conjectures by Dubrovin on Frobenius manifolds and mirror symmetry. Conjecture $\mathcal{O}$ has already been proved for many cases in [CL17, LMS19, HKLY, Ke, BFSS]. The Perron-Frobenius theory of nonnegative matrices reduces the proofs that Conjecture $\mathcal{O}$ holds for the homogeneous and the odd symplectic Grassmannian cases to be a graph-theoretic check. This is because Conjecture $\mathcal{O}$ is largely reminiscent of Perron-Frobenius Theory.

The small quantum cohomology is defined as follows. Let $(\alpha_i)_i$ be a basis of $H^*(X)$, the classical cohomology ring, and let $(\alpha_i^\vee)_i$ be the dual basis for the Poincaré pairing. The multiplication is given by

$$\alpha_i \star \alpha_j = \sum_{d \geqslant 0, k} c_{i,j}^{k,d} q^d \alpha_k$$

where $c_{i,j}^{k,d}$ are the 3-point, genus 0, Gromov-Witten invariants corresponding to the classes α_i, α_j, and $\alpha_k^\vee$. We will make use of the quantum Chevalley formula which is the multiplication of a hyperplane class h with another class α_j. The explicit quantum Chevalley formula is the key ingredient used to prove Property $\mathcal{O}$ holds.

1.6. **Sufficient Criterion for Property $\mathcal{O}$ to hold.** We recall the notion of the (oriented) quantum Bruhat graph of a Fano variety X. The vertices of this graph are the basis elements $\alpha_i \in H^\bullet(X) := QH^*(X)|_{q=1}$. There is an oriented edge $\alpha_i \to \alpha_j$ if the class α_j appears with positive coefficient (where we consider $q > 0$) in the quantum Chevalley multiplication

$h \star \alpha_i$ for some hyperplane class h. We say the graph is strongly connected if there are directed paths for α_i to α_j and α_j to α_i for any $\alpha_i, \alpha_j \in H^\bullet(X)$. If there is a path

$$\alpha_{i_1} \to \alpha_{i_2} \to \cdots \to \alpha_{i_k} \to \alpha_{i_1}$$

then we say this is a cycle with cycle length k. Using the Perron-Frobenius theory of non-negative matrices, Conjecture $\mathcal{O}$ reduces to a graph-theoretic check of the quantum Bruhat graph. The techniques involving Perron-Frobenius theory used by Li, Mihalcea, and Shifler in [LMS19] and Cheong and Li in [CL17] imply the following lemma:

Lemma 1.1. *If the following conditions hold for a Fano variety X:*

1. *The matrix representation of $\hat{c}_1$ is nonnegative,*
2. *The quantum Bruhat graph of X is strongly connected, and*
3. *There exists a set of cycles where the Great Common Divisor of the cycle lengths is r, the Fano index, in the quantum Bruhat graph of X,*

then Property $\mathcal{O}$ holds for X.

Remark 1.2. Part (2) of Conjecture $\mathcal{O}$ holds automatically if the Fano index is equal to one. That is, we don't need to verify Lemma 1.1 (3) for $\mathrm{IF}(1, 2; E)$.

1.7. **Combinatorial version of the quantum Bruhat graph and Conjecture $\mathcal{O}$.** Let X be a Fano variety. Let $\mathcal{B} := \{\alpha_i\}_{i \in I}$ denote a basis of the cohomology ring $H^*(X)$. Denote its first Chern class by

$$c_1(X) = a_1 \mathrm{Div}_1 + a_2 \mathrm{Div}_2 + \cdots + a_k \mathrm{Div}_k$$

where $\mathrm{Div}_i \in \mathcal{B}$ is a divisor class for each $1 \leqslant i \leqslant k$.

Definition 1.3. The combinatorial quantum Bruhat graph of X is define as follows. The vertices of this graph are the basis elements $\alpha_i \in H^*(X)$. The edges set is given as follows:

1. There is an oriented edge $\alpha_i \to \alpha_j$ if the class α_j appears with positive coefficient in the Chevalley multiplication $h \star \alpha_i$ for some hyperplane class h.
2. Let $\alpha_i = [X(i)]$ and $\alpha_j = [X(j)]$. There is an oriented edge $\alpha_i \to \alpha_j$
   (a) if $X(j)$ is an irreducible component of $\Gamma_d(X(i))$ where $d = (d_1, \cdots, d_k)$,
   (b) and

$$\dim(X(j)) - \dim(X(i)) = a_1 d_1 + a_2 d_2 + \cdots + a_k d_k - 1.$$

Lemma 1.1 leads us to naturally consider the following combinatorial formulation of Conjecture $\mathcal{O}$.

Definition 1.4. Combinatorial Property $\mathcal{O}$ holds if the following conditions are satisfied:

1. There is a basis with respect to which the associated matrix of $\hat{c}_1$ is nonnegative[1],
2. The combinatorial quantum Bruhat graph is strongly connected and the Greatest Common Divisor of the cycle lengths is $r = \mathrm{GCD}(a_1, a_2, \cdots, a_k)$.

The purpose of Proposition 1.5 is to assert that the Combinatorial versions of the quantum Bruhat graph and Conjecture $\mathcal{O}$ correspond with their geometric analogue.

Proposition 1.5 ([LMS19, CL17])**.** *The combinatorial quantum Bruhat graphs (Combinatorial Property $\mathcal{O}$) and the quantum Bruhat graphs (Conjecture $\mathcal{O}$) correspond for homogeneous G/P and the odd symplectic Grassmannian.*

[1]General Fano manifolds may not satisfy this property. For instance, the blowup of $\mathbb{P}^2$ at a point does not satisfy this property.

1.8. **Organization of the main results.** The manuscript has three objectives. The first is to compute the curve neighborhoods of Schubert varieties in IF. This is done in Section 4. We study curve neighborhoods in the context of lattices in Section 5. In Section 6 we study the combinatorial analogues of the quantum Bruhat graphs and Conjecture $\mathcal{O}$. We conclude by showing that Combinatorial Property $\mathcal{O}$ holds which motivate futher study of IF.

2. Notations and Definitions

2.1. **A small family of odd symplectic partial flag manifolds.** Let $n \geqslant 2$ and $E := \mathbb{C}^{2n+1}$ be an odd-dimensional complex vector space. An *odd symplectic form* ω on E is a skew-symmetric bilinear form of maximal rank (i.e. with kernel of dimension 1). It will be convenient to extend the form ω to a (nondegenerate) symplectic from $\tilde{\omega}$ on an even-dimensional space $\widetilde{E} \supset E$, and to identify $E \subset \widetilde{E}$ with a coordinate hyperplane $\mathbb{C}^{2n+1} \subset \mathbb{C}^{2n+2}$.

For that, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}, \mathbf{e}_{\overline{n+1}}, \ldots, \mathbf{e}_{\bar{2}}, \mathbf{e}_{\bar{1}}\}$ be the standard basis of $\widetilde{E} := \mathbb{C}^{2n+2}$, where $\bar{i} = 2n+3-i$. Consider $\tilde{\omega}$ to be the symplectic form on $\widetilde{E}$ defined by

$$\tilde{\omega}(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,\bar{j}} \text{ for all } 1 \leqslant i \leqslant j \leqslant \bar{1}.$$

The form $\tilde{\omega}$ restricts to the degenerate skew-symmetric form ω on

$$E = \mathbb{C}^{2n+1} = \langle \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{2n+1} \rangle$$

such that the kernel $\ker \omega$ is generated by $\mathbf{e}_1$. Then

$$\omega(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,\bar{j}} \text{ for all } 1 \leqslant i \leqslant j \leqslant \bar{2}.$$

Let $F \subset E$ denote the $2n$-dimensional vector space with basis $\{\mathbf{e}_2, \mathbf{e}_3, \cdots, \mathbf{e}_{2n+1}\}$.

The odd symplectic partial flag manifold we are considering is

$$\mathrm{IF}(1,2;E) := \{V_1 \subset V_2 \subset E : \dim V_i = i \text{ and } \omega(x,y) = 0 \text{ for any } x, y \in V_i\}.$$

The restriction of ω to F is non-degenerate, hence we can see the odd symplectic Grassmannian as intermediate space

$$\mathrm{IF}(1;F) \subset \mathrm{IF}(1,2;E) \subset \mathrm{IF}(1,2;\widetilde{E}), \tag{1}$$

sandwiched between two odd symplectic flag manifolds. This and the more general "odd symplectic partial flag varieties" have been studied in [Mih07, Pec13, MS19, LMS19, PS22]. In particular, Mihai showed that IF is a smooth Schubert variety in $\mathrm{IF}(1,2;\widetilde{E})$.

2.2. **The odd symplectic group.** Proctor's *odd symplectic group* (see [Pro88]) is the subgroup of $\mathrm{GL}(E)$ which preserves the odd symplectic form ω:

$$\mathrm{Sp}_{2n+1}(E) := \{g \in \mathrm{GL}(E) \mid \omega(g \cdot u, g \cdot v) = \omega(u,v), \forall u, v \in E\}.$$

Let $\mathrm{Sp}_{2n}(F)$ and $\mathrm{Sp}_{2n+2}(\widetilde{E})$ denote the symplectic groups which respectively preserve the symplectic forms $\omega_{|F}$ and $\tilde{\omega}$. Then with respect to the decomposition $E = F \oplus \ker \omega$ the elements of the odd symplectic group $\mathrm{Sp}_{2n+1}(E)$ are matrices of the form

$$\mathrm{Sp}_{2n+1}(E) = \left\{ \begin{pmatrix} \lambda & a \\ 0 & S \end{pmatrix} \mid \lambda \in \mathbb{C}^*, a \in \mathbb{C}^{2n}, S \in \mathrm{Sp}_{2n}(F) \right\}.$$

The symplectic group $\mathrm{Sp}_{2n}(F)$ embeds naturally into $\mathrm{Sp}_{2n+1}(E)$ by $\lambda = 1$ and $a = 0$, but $\mathrm{Sp}_{2n+1}(E)$ is *not* a subgroup of $\mathrm{Sp}_{2n+2}(\widetilde{E})$.[2] Mihai showed in [Mih07, Prop 3.3] that there is a surjection $P \to \mathrm{Sp}_{2n+1}(E)$ where $P \subset \mathrm{Sp}_{2n+2}(\widetilde{E})$ is the parabolic subgroup which preserves $\ker\omega$, and the map is given by restricting $g \mapsto g_{|E}$. Then the Borel subgroup $B_{2n+2} \subset \mathrm{Sp}_{2n+2}(\widetilde{E})$ of upper triangular matrices restricts to the (Borel) subgroup $B \subset \mathrm{Sp}_{2n+1}(E)$. Similarly, the maximal torus

$$T_{2n+2} := \{\mathrm{diag}(t_1, \cdots, t_{n+1}, t_{n+1}^{-1}, \cdots, t_1^{-1}) : t_1, \cdots, t_{n+1} \in \mathbb{C}^*\} \subset B_{2n+2}$$

restricts to the maximal torus

$$T = \{\mathrm{diag}(t_1, \cdots, t_{n+1}, t_{n+1}^{-1}, \cdots, t_2^{-1}) : t_1, \cdots, t_{n+1} \in \mathbb{C}^*\} \subset B.$$

Later on we will also require notation for subgroups of $\mathrm{Sp}_{2n}(F)$, viewed as a subgroup of $\mathrm{Sp}_{2n+1}(E)$. We denote by $B_{2n} \subset B$ the Borel subgroup of upper-triangular matrices in $\mathrm{Sp}_{2n}(F)$ and by T_{2n} the maximal torus

$$T_{2n} = \{\mathrm{diag}(1, t_2, \cdots, t_{n+1}, t_{n+1}^{-1}, \cdots, t_2^{-1}) : t_2, \cdots, t_{n+1} \in \mathbb{C}^*\} \subset B_{2n}.$$

Mihai showed that the odd symplectic group $\mathrm{Sp}_{2n+1}(E)$ acts on IF with three orbits:

$$\begin{aligned} X^\circ &= \{V \in \mathrm{IF} \mid \mathbf{e}_1 \notin V_2\} \text{ the open orbit.}\\ Z_2 &= \{V \in \mathrm{IF} \mid \mathbf{e}_1 \in V_2 \backslash V_1\}.\\ Z_1 &= \{V \in \mathrm{IF} \mid \mathbf{e}_1 \in V_1\} \text{ the closed orbit.} \end{aligned}$$

The closed orbit Z_1 is isomorphic to $\mathrm{IF}(1, F)$ via the map $V \mapsto V \cap F$.

2.3. **The Weyl group of Sp_{2n+2} and odd symplectic minimal representatives.** There are many possible ways to index the Schubert varieties of isotropic flag manifolds. Here we recall an indexation using signed permutations.

Consider the root system of type C_{n+1} with positive roots

$$R^+ = \{t_i \pm t_j \mid 1 \leqslant i < j \leqslant n+1\} \cup \{2t_i \mid 1 \leqslant i \leqslant n+1\}$$

and the subset of simple roots

$$\Delta = \{\alpha_i := t_i - t_{i+1} \mid 1 \leqslant i \leqslant n\} \cup \{\alpha_{n+1} := 2t_{n+1}\}.$$

The associated Weyl group W is the hyperoctahedral group consisting of *signed permutations*, i.e. permutations w of the elements $\{1, \cdots, n+1, \overline{n+1}, \cdots, \overline{1}\}$ satisfying $w(\overline{i}) = \overline{w(i)}$ for all $w \in W$. For $1 \leqslant i \leqslant n$ denote by s_i the simple reflection corresponding to the root $t_i - t_{i+1}$ and s_{n+1} the simple reflection of $2t_{n+1}$. In particular, if $1 \leqslant i \leqslant n$ then $s_i(i) = i+1$, $s_i(i+1) = i$, and $s_i(j)$ is fixed for all other j. Also, $s_{n+1}(n+1) = \overline{n+1}$, $s_{n+1}(\overline{n+1}) = n+1$, and $s_{n+1}(j)$ is fixed for all other j.

Each subset $I := \{i_1 < \ldots < i_r\} \subset \{1, \ldots, n+1\}$ determines a parabolic subgroup $P := P_I \leqslant \mathrm{Sp}_{2n+2}(\widetilde{E})$ with Weyl group $W_P = \langle s_i \mid i \neq i_j \rangle$ generated by reflections with indices *not* in I. Let $\Delta_P := \{\alpha_{i_s} \mid i_s \notin \{i_1, \ldots, i_r\}\}$ and $R_P^+ := \mathrm{Span}_{\mathbb{Z}} \Delta_P \cap R^+$; these are the positive roots of P. Let $\ell \colon W \to \mathbb{N}$ be the length function and denote by W^P the set of minimal length representatives of the cosets in W/W_P. The length function descends to W/W_P by $\ell(uW_P) = \ell(u')$ where $u' \in W^P$ is the minimal length representative for the coset uW_P. We have a natural ordering

$$1 < 2 < \cdots < n+1 < \overline{n+1} < \cdots < \overline{1},$$

[2] However, Gelfand and Zelevinsky [GZ84] defined another group $\widetilde{\mathrm{Sp}}_{2n+1}$ closely related to Sp_{2n+1} such that $\mathrm{Sp}_{2n} \subset \widetilde{\mathrm{Sp}}_{2n+1} \subset \mathrm{Sp}_{2n+2}$.

which is consistent with our earlier notation $\bar{i} := 2n + 3 - i$.

Let P be the parabolic obtained by excluding the reflections s_1 and s_2. Then the minimal length representatives W^P have the form $(w(1)|w(2)|w(3) < \cdots < w(n) \leqslant n+1)$. Since the last $n-1$ labels are determined from the first 2 labels, we will identify an element in W^P with $(w(1)|w(2))$.

Example 2.1. The reflection $s_{t_1+t_2}$ is given by the signed permutation

$$s_{t_1+t_2}(1) = \bar{2}, s_{t_1+t_2}(2) = \bar{1}, \text{ and } s_{t_1+t_2}(i) = i \text{ for all } 3 \leqslant i \leqslant n+1.$$

The minimal length representative of $s_{t_1+t_2}W^P$ is $(\bar{2}|\bar{1})$.

2.4. **Schubert Varieties in even and odd symplectic partial flag manifolds.** Recall that the even symplectic partial flag manifold $X^{ev} = \mathrm{IF}(1, 2; \tilde{E})$ is a homogeneous space Sp_{2n+2}/P, where P is the parabolic subgroup generated by the simple reflections s_i with $i \neq 1, 2$. For each $w \in W^P$ let $X^{ev}(w)^\circ := B_{2n+2}wB_{2n+2}/P$ be the *Schubert cell*. This is isomorphic to the space $\mathbb{C}^{\ell(w)}$. Its closure $X^{ev}(w) := \overline{X^{ev}(w)^\circ}$ is the *Schubert variety*. We might occasionally use the notation $X^{ev}(wW_P)$ if we want to emphasize the corresponding coset, or if w is not necessarily a minimal length representative. Recall that the Bruhat ordering can be equivalently described by $v \leqslant w$ if and only if $X^{ev}(v) \subset X^{ev}(w)$. Set

$$w_0 = (\bar{2}|\bar{3})$$

this is an element in W. Recall that the odd symplectic Borel subgroup is $B = B_{2n+2} \cap \mathrm{Sp}_{2n+1}$. The following results were proved by Mihai [Mih07, §4].

Remark 2.2. Here w_0 is the longest element for IF which is different than the longest element for $\mathrm{IF}(1, 2; \tilde{E})$.

Proposition 2.3. *(a) The natural embedding $\iota : X = \mathrm{IF} \hookrightarrow X^{ev} = \mathrm{IF}(1, 2; \tilde{E})$ identifies* IF *with the (smooth) Schubert subvariety*

$$X^{ev}(w_0W_P) \subset \mathrm{IF}(1, 2; \tilde{E}).$$

(b) The Schubert cells (i.e. the B_{2n+2}-orbits) in $X^{ev}(w_0)$ coincide with the B-orbits in IF*. In particular, the B-orbits in* IF *are given by the Schubert cells $X^{ev}(w)^\circ \subset \mathrm{IF}(1, 2; \tilde{E})$ such that $w \leqslant w_0$.*

We discuss Schubert cells or varieties in the odd symplectic case. For each $w \leqslant w_0$ such that $w \in W^P$, we denote by $X(w)^\circ$, and $X(w)$, the Schubert cell, respectively the Schubert variety in IF. The same Schubert variety $X(w)$, but regarded in the even symplectic partial flag manifold is denoted by $X^{ev}(w)$. For further use we note that IF has complex dimension $\ell(\bar{2}|\bar{3}) = 4n-6$, $\mathrm{IF}(1, 2; \tilde{E})$ has complex dimension $\ell(\bar{1}|\bar{2}) = 4n-4$, and IF has codimension 2 in $\mathrm{IF}(1, 2; \tilde{E})$. Further, a Schubert variety $X(w)$ in IF is included in the closed Sp_{2n+1}-orbit Z_1 of if and only if it has a minimal length representative $w \leqslant w_0$ such that $w(1) = 1$.

Define the set $W^{odd} := \{w \in W \mid w \leqslant w_0\}$ and call its elements *odd symplectic permutations*. The set W^{odd} consists of permutations $w \in W$ such that $w(j) \neq \bar{1}$ for any $1 \leqslant j \leqslant n+1$ [Mih07, Prop. 4.16].

2.5. **Divisors and the first Chern class.**

Lemma 2.4. *The odd symplective flag manifold* IF *has two divisor classes. If $n = 2$ then*

$$[X(Div_1)] := [X(\bar{3} \mid \bar{2})] \quad \textit{and} \quad [X(Div_2)] := [X(\bar{2} \mid 3)].$$

If $n > 2$ then

$$[X(Div_1)] := [X(\bar{3} \mid \bar{2})] \quad \textit{and} \quad [X(Div_2)] := [X(\bar{2} \mid \bar{4})].$$

Lemma 2.5. *The first Chern class of* IF *is*

$$c_1(\mathrm{IF}) = 2 \cdot [X(Div_1)] + (2n-1) \cdot [X(Div_2)].$$

3. The Moment Graph

Sometimes called the GKM graph, the *moment graph* of a variety with an action of a torus T has a vertex for each T-fixed point, and an edge for each 1-dimensional torus orbit. The description of the moment graphs for flag manifolds is well known, and it can be found e.g in [Kum02, Ch. XII]. In this section we consider the moment graphs for $X = \mathrm{IF}(1,2;E) \subset X^{ev} = \mathrm{IF}(1,2;\tilde{E})$. As before let $P \subset \mathrm{Sp}_{2n+2}$ be the maximal parabolic for X^{ev}. Recall that we will identify an element in W^P with $(w(1)|w(2))$.

3.1. **Moment graph structure of** $\mathrm{IF}(1,2;\tilde{E})$**.** The moment graph of X^{ev} has a vertex for each $w \in W^P$, and an edge $w \to ws_\alpha$ for each

$\alpha \in R^+ \backslash R_P^+ = \{t_i - t_j \mid 1 \leqslant i \leqslant 2, i < j \leqslant n+1\} \cup \{t_i + t_j, 2t_i \mid 1 \leqslant i \leqslant 2, 1 \leqslant i < j \leqslant n+1\}.$

Geometrically, this edge corresponds to the unique torus-stable curve $C_\alpha(w)$ joining w and ws_α. The curve $C_\alpha(w)$ has degree $d = (d_1, d_2)$, where $\alpha^\vee + \Delta_P^\vee = d_1\alpha_1^\vee + d_2\alpha_2^\vee + \Delta_P^\vee$. In the next section we classify the positive roots by their degree. In order to perform curve neighborhood calculations we will we give precise combinatorial description of the moment graph.

Definition 3.1. Define the following to describe moment graph combinatorics.

(1) Define the following four sets which partitions $R^+ \backslash R_P^+$.
   (a) $R^+_{(1,0)} = \{t_1 - t_2\}$;
   (b) $R^+_{(0,1)} = \{t_2 \pm t_j \mid 3 \leqslant j \leqslant n+1\} \cup \{2t_2\}$;
   (c) $R^+_{(1,1)} = \{t_1 \pm t_j : 3 \leqslant j \leqslant n+1\} \cup \{2t_1\}$;
   (d) $R^+_{(1,2)} = \{t_1 + t_2\}$.
(2) A *chain of degree* d is a path in the (unoriented) moment graph where the sum of edge degrees equals d. We will often use that notation $uW_P \xrightarrow{d} vW_P$ to denote such a path.

In the next lemma we give a formula for the degree d of a chain which is useful to calculate curve neighborhoods. In particular, we will see that the degree of a chain is determined by summing the weights of the edges traversed in the moment graph.

Lemma 3.2. *Let* $u, v \in W^P$ *be connected by a degree* d *chain*

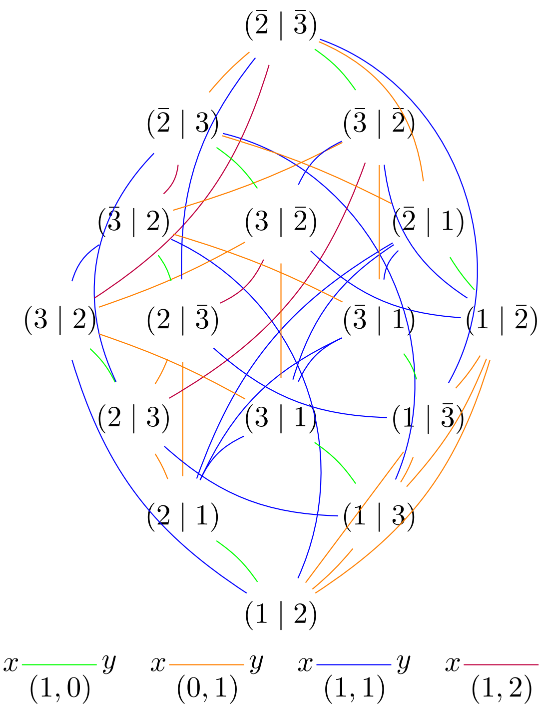
$$(uW_P \xrightarrow{d} vW_P) = (uW_P \to us_{\alpha_1}W_P \to \cdots \to us_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_t}W_P)$$

where $vW_P = us_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_t}W_P$ *and the* α_i *are in* $R^+ \backslash R_P^+$*. Then:*

$$\begin{aligned} d = \left(\#\left\{\alpha_i \in R^+_{(1,0)}\right\}, \#\left\{\alpha_i \in R^+_{(0,1)}\right\}\right) &+ \left(\#\left\{\alpha_i \in R^+_{(1,1)}\right\}, \#\left\{\alpha_i \in R^+_{(1,1)}\right\}\right) \\ &+ \left(\#\left\{\alpha_i \in R^+_{(1,2)}\right\}, 2 \cdot \#\left\{\alpha_i \in R^+_{(1,2)}\right\}\right). \end{aligned}$$

3.2. **Moment graph structure of** IF**.** The moment graph of IF is the full subgraph of $\mathrm{IF}(1,2;2n+2)$ determined by the vertices $w \in W^P \cap W^{odd}$. Notice that the orbits of T and T_{2n+2} coincide, therefore we do not distinguish between the moment graphs for these tori. See Figure 1.

FIGURE 1. The moment graph for IF when $n = 2$.



4. CURVE NEIGHBORHOODS

The main result of this section is Theorem 4.3 which states all curve neighborhoods of Schubert varieties IF. We will define a curve neighborhood in Definition 4.1 and Proposition 4.2 will state the combinatorial equivalent version. Then Lemmas 4.4 and 4.5 are used to prove the main result in Theorem 4.3.

Let X be a Fano variety. Let $d \in H_2(X, \mathbb{Z})$ be an effective degree. Recall that the moduli space of genus 0, degree d stable maps with two marked points $\overline{\mathcal{M}}_{0,2}(X, d)$ is endowed with two evaluation maps $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{0,2}(X, d) \to X$, $i = 1, 2$ which evaluate stable maps at the i-th marked point.

Definition 4.1. Let $\Omega \subset X$ be a closed subvariety. The *curve neighborhood* of Ω is the subscheme

$$\Gamma_d(\Omega) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}\,\Omega) \subset X$$

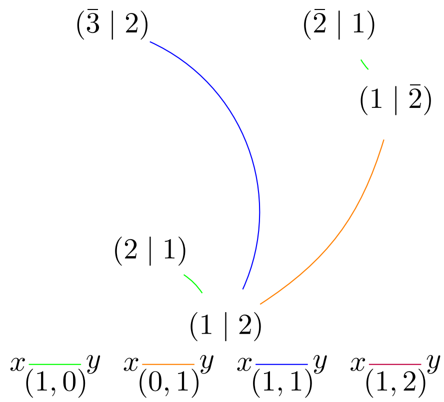
endowed with the reduced scheme structure.

The next proposition gives a combinatorial formulation of curve neighborhoods in terms of the moment graph. See Figure 2 for an example of an application of Proposition 4.2.

Proposition 4.2 ([BM15])**.** *Let $w \in W^P \cap W^{odd}$. In the moment graph of $X = \mathrm{IF}$, let $\{v^1, \cdots, v^s\}$ be the maximal vertices (for the Bruhat order) which can be reached from any $u \leqslant w$ using a chain of degree d or less. Then $\Gamma_d(X(w)) = X(v^1) \cup \cdots \cup X(v^s)$.*

Proof. Let $Z_{w,d} = X(v^1) \cup \cdots \cup X(v^s)$. Let $v := v^i \in Z_{w,d}$ be one of the maximal T-fixed points. By the definition of v and the moment graph there exists a chain of T-stable rational curves of degree less than or equal to d joining $u \leqslant w$ to v. It follows that there exists a

Figure 2. In this figure we calculate a few curve neighborhoods of the Schubert point $(1|2)$ for $n = 2$ as an example of Proposition 4.2.



$$\Gamma_{(1,0)}(X(1|2)) = X(2|1) \qquad \Gamma_{(0,1)}(X(1|2)) = X(1|\bar{2}) \qquad \Gamma_{(1,1)}(X(1|2)) = X(\bar{3}|2) \cup X(\bar{2}|1)$$

degree d stable map joining $u \leqslant w$ to v. Therefore $v \in \Gamma_d(X(w))$, thus $X(v) \subset \Gamma_d(X(w))$, and finally $Z_{w,d} \subset \Gamma_d(X(w))$.

For the converse inclusion, let $v \in \Gamma_d(X(w))$ be a T-fixed point. By [MM18, Lemma 5.3] there exists a T-stable curve joining a fixed point $u \in X(w)$ to v. This curve corresponds to a path of degree d or less from some $u \leqslant w$ to v in the moment graph of $\mathrm{IG}(k, 2n+1)$. By maximality of the v^i it follows that $v \leqslant v^i$ for some i, hence $v \in X(v^i) \subset Z_{w,d}$, which completes the proof. □

The next theorem is the main result of this section that states the precise calculations of curve neighborhoods of Schubert varieties in IF. The result follows from Lemmas 4.4 and 4.5 stated below.

Theorem 4.3. *The following curve neighborhood calculations hold.*

(1) $\Gamma_{(d_1 \geqslant 1, 0)}(X(a|b)) = \begin{cases} X(a|b); a > b \\ X(b|a); a < b \end{cases}$

(2) $\Gamma_{(0, d_2 \geqslant 1)}(X(a|b)) = \begin{cases} X(a|\bar{3}); a \in \{2, \bar{2}\} \\ X(a|\bar{2}); a \notin \{2, \bar{2}\} \end{cases}$

(3) $\Gamma_{(d_1 \geqslant 1, 1)}(X(a|b)) = \begin{cases} X(\bar{3}|2) \cup X(\bar{2}|1); (a|b) \in \{(1|2), (2|1)\} \\ X(\bar{2}|\max\{a,b\}); 1 \leqslant a, b \leqslant \bar{3} \textit{ and } (a|b) \notin \{(1|2), (2|1)\} \\ X(\bar{2}|\bar{3}); \bar{2} \in \{a, b\} \end{cases}$

(4) $\Gamma_{(d_1 \geqslant 1, d_2 \geqslant 2)}(X(a|b)) = X(\bar{2}|\bar{3})$

Proof. For case (1), if $a > b$ then $(a|b) \cdot s_1 = (a|b)$ and if $a < b$ then $(a|b) \cdot s_1 = (b|a)$.

For case (2) we will need to check four subcases. Notice that we are multiplying $(a|b)$ by reflections of the form $t_2 \pm t_j$ for $3 \leqslant j \leqslant n+1$ or $2t_2$. In particular, a remains fixed so $\Gamma_{(0,1)}(X(a|b)) = \Gamma_{(0,d_2 \geqslant 1)}(X(a|b))$. If $a \in \{2, \bar{2}\}$ and $b \notin \{3, \bar{3}\}$ then $(a|b) \cdot s_{t_2+t_3} = (a|\bar{3})$.

If $a \in \{2, \bar{2}\}$ and $b \in \{3, \bar{3}\}$ then $(a|b) \cdot s_{2t_2} = (a|\bar{3})$. If $a \notin \{2, \bar{2}\}$ and $b \notin \{2, \bar{2}\}$ then $(a|b) \cdot s_{t_2+t_3} = (a|\bar{2})$. If $a \notin \{2, \bar{2}\}$ and $b \in \{2, \bar{2}\}$ then $(a|b) \cdot s_{2t_2} = (a|\bar{2})$. In each case, this is the vertex with the greatest length that can be reached by and edge of degree $(0,1)$ from $(a|b)$.

For case (3), first notice that $\Gamma_{(1,1)}(X(a|b)) = \Gamma_{(d_1 \geqslant 1,1)}(X(a|b))$ since the reflection s_1 only interchanges $(a|b)$ and there is not an edge in the moment graph of IF with degree $(2,1)$. Case (3) follows by Lemma 4.5.

For case (4), notice that $\Gamma_{(1,2)}(X(a|b)) = \Gamma_{(d_1 \geqslant 1, d_2 \geqslant 1)}(X(a|b))$. By case (3), we see that $\Gamma_{(1,1)}(X(a|b))$ has a component of the form $X(\bar{2}|b)$. The result for case (4) follows by an application of case (2). The result follows. ☐

In the next lemma we state the possible chains in the moment graph of IF of degree less than or equal to $(1,1)$.

Lemma 4.4. *Let $(a|b) \in W^{odd}$. Then we have the following chains in the moment graph of* IF.

(1) For chains traversing an edge of degree $(1,0)$ followed by an edge of degree $(0,1)$, we have the following:

$$(a|b) \xrightarrow{(1,0)} (b|a) \xrightarrow{(0,1)} (b|h)$$

where $h \notin \{b, \bar{b}\}$.

(2) For chains traversing an edge of degree $(0,1)$ followed by an edge of degree $(1,0)$ we have the following:

$$(a|b) \xrightarrow{(0,1)} (a|h) \xrightarrow{(1,0)} (h|a)$$

where $h \notin \{a, \bar{a}\}$. Notice that if $a = \bar{2}$ then permutation length decreases on the second step.

(3) For chains traverse an edge of degree $(1,1)$ we have the following:

$$(a|b) \xrightarrow{(1,1)} (h|b)$$

where $h \notin \{b, \bar{b}\}$.

Proof. We will prove each case. For case (1), following an edge of degree $(1,0)$ from $(a|b)$ results in $(b|a)$ by Lemma 3.2. Then following an edge of degree $(0,1)$ from $(b|a)$ results in $(b|h)$ where $h \notin \{b, \bar{b}\}$ by Lemma 3.2.

For case (2), following an edge of degree $(0,1)$ from $(a|b)$ results in $(a|h)$ where $h \notin \{a, \bar{a}\}$ by Lemma 3.2. Then following an edge of degree $(1,0)$ from $(a|h)$ results in $(h|a)$ by Lemma 3.2.

For case (3), following an edge of degree $(1,1)$ from $(a|b)$ results in $(h|b)$ where $h \notin \{b, \bar{b}\}$ by Lemma 3.2. The result follows. ☐

In this lemma, we calculate the curve neighborhood $\Gamma_{(d_1 \geqslant 1,1)}(X(a|b))$ for any $(a|b) \in W^P$.

Lemma 4.5. *First note the $\Gamma_{(d_1 \geqslant 1,1)}(X(a|b)) = \Gamma_{(1,1)}(X(a|b))$ for any $(a|b) \in W^{odd}$ since right multiplication by s_1 on $w \in W$ interchanges $w(1)$ and $w(2)$. The curve neighborhood $\Gamma_{(1,1)}(X(a|b))$ is given by one of the following.*

(1) Suppose $a \notin \{2, \bar{2}\}$ and $b \notin \{2, \bar{2}\}$.

(a) If $a < b$ then

$$\Gamma_{(1,1)}(X(a|b)) = X(\bar{2}|b).$$

(b) *If* $a > b$ *then*

$$\Gamma_{(1,1)}(X(a|b)) = X(\bar{2}|a).$$

(2) *Suppose* $a \notin \{2, \bar{2}\}$ *and* $b \in \{2, \bar{2}\}$.

(a) *If* $b = \bar{2}$ *then*

$$\Gamma_{(1,1)}(X(a|b)) = X(\bar{2}|\bar{3}).$$

(b) *If* $a > 2$ *and* $b = 2$ *then*

$$\Gamma_{(1,1)}(X(a|b)) = X(\bar{2}|a).$$

(c) *If* $a = 1$ *and* $b = 2$ *then*

$$\Gamma_{(1,1)}(X(1|2)) = X(\bar{3}|2) \cup X(\bar{2}|1).$$

(3) *Suppose* $a \in \{2, \bar{2}\}$ *and* $b \notin \{2, \bar{2}\}$.

(a) *If* $a = \bar{2}$ *then*

$$\Gamma_{(1,1)}(X(a|b)) = X(\bar{2}|\bar{3}).$$

(b) *If* $a = 2$ *and* $b > 2$ *then*

$$\Gamma_{(1,1)}(X(a|b)) = X(\bar{2}|b).$$

(c) *If* $a = 2$ *and* $b = 1$ *then*

$$\Gamma_{(1,1)}(X(2|1)) = X(\bar{3}|2) \cup X(\bar{2}|1).$$

Proof. The result follows from Lemma 4.4 and the observation that the pair $\{(\bar{3}|2), (\bar{2}|1)\}$ is incomparable in the Bruhat order. The result follows. $\square$

5. Lattices

Let X be a Fano variety containing the subvariety Ω. It is interesting to ask if the set $\{\Gamma_d(\Omega)\}_d$ forms a (distributive) lattice where $\leqslant$ is defined by inclusion of varieties. We will show that

$$\{\Gamma_{(d_1,d_2)}(X(a|b))\}_{(d_1,d_2)\geqslant(0,0)}$$

is a distributive lattice for any $(a|b) \in W^{odd}$. We will review the definition of a (distributive) lattice and two particular lattices next.

Definition 5.1. We will define lattices, distributive lattices, and two useful lattices.

(1) A partially ordered set $(L, \leqslant)$ is a lattice if the following two conditions hold.
   (a) if for any $a, b \in L$ there is a unique element denoted by $a \vee b \in L$ such that
   - $a \leqslant a \vee b$ and $b \leqslant a \vee b$
   - and if there is a $c \in L$ such that $a \leqslant c$ and $b \leqslant c$ then $a \vee b \leqslant c$.

   (b) if for any $a, b \in L$ there is a unique element denoted by $a \wedge b \in L$ such that
   - $a \geqslant a \vee b$ and $b \geqslant a \vee b$
   - and if there is a $c \in L$ such that $a \geqslant c$ and $b \geqslant c$ then $a \wedge b \geqslant c$.

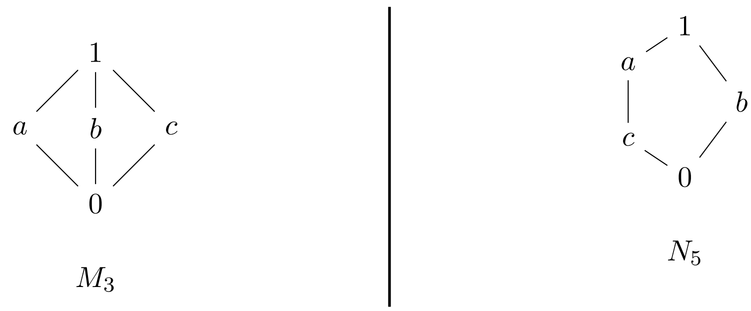
(2) A Lattice $(L, \leqslant)$ is a distributive lattice if

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for any $a, b, c \in L$.

(3) Define the lattice M_3 and N_5 as follows:

Figure 3. M_3 and N_5.



M_3 N_5

(a) $M_3 = \{0, a, b, c, 1\}$ with $\leqslant$ defined by
- $0 < a, b, c$,
- $a, b, c < 1$, and
- a, b, and c are incomparable.

(b) $N_5 = \{0, a, b, c, 1\}$ with $\leqslant$ defined by
- $0 < c < a < 1$
- $0 < b < 1$
- b is incomparable with both a and c.

See Figure 3.

The next lemma gives a precise description of when a lattice is distributive.

Lemma 5.2. [Grä98, Chapter 2, Theorem 1] *A lattice is distributive if and only if it does not have a sub-lattice isomorphic to M_3 nor N_5.*

We are ready to state the main result of the section.

Theorem 5.3. *The set $L = \{\Gamma_{(d_1,d_2)}(X(a|b))\}_{(d_1,d_2)\geqslant(0,0)}$ where $\leqslant$ is defined by inclusions of varieties is a distributive lattice.*

Proof. Figure 4 lists each lattice for each case except the trivial case $(a|b) = (\bar{2}|\bar{3})$. Furthermore, $(L, \leqslant)$ is distributive because neither M_3 nor N_5 is isomorphic to a sub-lattice of L. $\square$

6. Combinatorial Property $\mathcal{O}$

We begin by recalling the definitions of the combinatorial quantum Bruhat graph and Combinatorial Conjecuture $\mathcal{O}$. Let X be a Fano variety. Let $\mathcal{B} := \{\alpha_i\}_{i\in I}$ denote a basis of the cohomology ring $H^*(X)$. Denote its first Chern class by

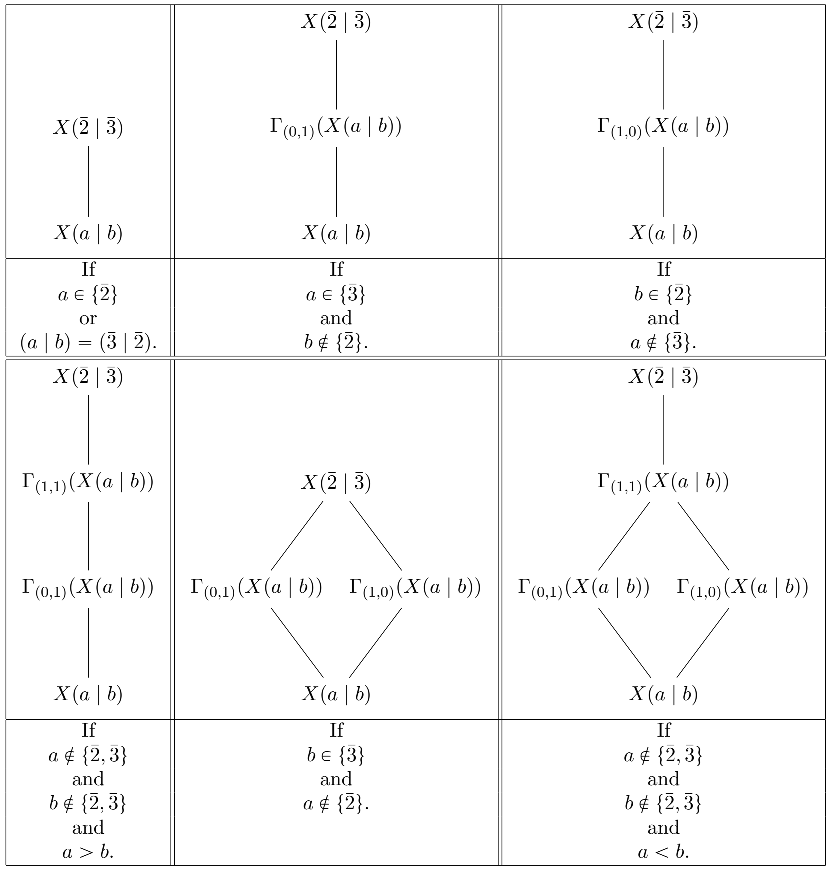
$$c_1(X) = a_1\mathrm{Div}_1 + a_2\mathrm{Div}_2 + \cdots + a_k\mathrm{Div}_k$$

where $\mathrm{Div}_i \in \mathcal{B}$ is a divisor class for each $1 \leqslant i \leqslant k$.

Definition 6.1. The combinatorial quantum Bruhat graph of X is defined as follows. The vertices of this graph are the basis elements $\alpha_i \in H^*(X)$. The edge set is given as follows:

(1) There is an oriented edge $\alpha_i \to \alpha_j$ if the class α_j appears with positive coefficient in the Chevalley multiplication $h \star \alpha_i$ for some hyperplane class h.
(2) Let $\alpha_i = [X(i)]$ and $\alpha_j = [X(j)]$. There is an oriented edge $\alpha_i \to \alpha_j$
   (a) $X(j)$ is an irreducible component of $\Gamma_d(X(i))$ where $d = (d_1, \cdots, d_k)$,

FIGURE 4. The table contains the lattices $(\{\Gamma_{(d_1,d_2)}(X(a|b))\}_{(d_1,d_2)\geqslant(0,0)}, \leqslant)$ in IF for each possible case except the trivial case when $(a|b) = (\bar{2}|\bar{3})$.



(b) and

$$\dim(X(j)) - \dim(X(i)) = a_1d_1 + a_2d_2 + \cdots + a_kd_k - 1.$$

Lemma 1.1 leads us to naturally consider the following combinatorial formulation of Conjecture $\mathcal{O}$.

Definition 6.2. Combinatorial Property $\mathcal{O}$ holds if the combinatorial quantum Bruhat graph is strongly connected and the Greatest Common Divisor of the cycle lengths is $r = \mathrm{GCD}(a_1, a_2, \cdots, a_k)$.

6.1. **Results for** IF**.** We begin this section by describing the combinatorial quantum Bruhat graph for IF which specializes Definition 6.1.

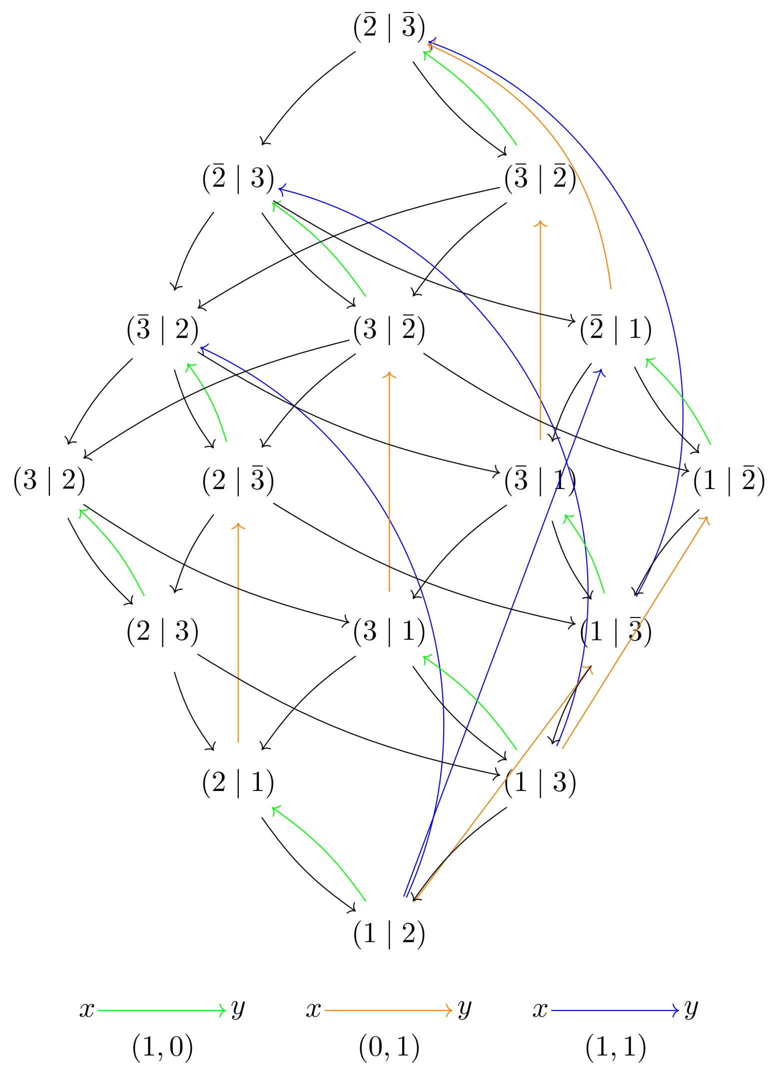
Proposition 6.3. *Let there be a vertex for each $w \in W^{odd}$. Let $u, v \in W^{odd}$. The combinatorial quantum Bruhat graph $\mathcal{G}_n$ can be created by:*

(1) There is an arrow $u \to v$ if $v \leqslant u$ and $\ell(v) = \ell(u) - 1$;

(2) Draw an arrow $u \xrightarrow{d} v$ if:
- *$\Gamma_d(X(u)) = X(v_1) \cup \cdots \cup X(v) \cup \cdots \cup X(v_s)$;*
- *$v \nleqslant v_i$ for any $1 \leqslant i \leqslant s$;*

- $\ell(v) - \ell(u) = 2d_1 + (2n-1)d_2 - 1$.

FIGURE 5. Combinatorial quantum Bruhat graph of IF for $n = 2$. Notice that the edge joining $(1|2)$ and $(\bar{2}|1)$ is **not** in the moment graph.



Remark 6.4. Observe that the edge $(1|2) \overset{(1,1)}{\to} (\bar{2}|1)$ in the combinatorial quantum Bruhat graph is **not** present in the moment graph. There is no known example of an edge appearing in the (geometric) quantum Bruhat graph that does not appear in the moment graph.

We are ready to state and prove the main result of this section.

Theorem 6.5. *Combinatorial Property $\mathcal{O}$ holds for* IF*.*

Proof. We begin the proof by showing that combinatorial quantum Bruhat graph is strongly connected. First, there is a path from $(1|2)$ to $(\bar{2}|\bar{3})$ in the combinatorial quantum Bruhat graph given by

$$(1|2) \overset{(1,1)}{\to} (\bar{2}|1) \overset{(0,1)}{\to} (\bar{2}|\bar{3})$$

since

$$\Gamma_{(1,1)}(X(1|2)) = X(\bar{2}|1), \Gamma_{(0,1)}(X(\bar{2}|1)) = X(\bar{2}|\bar{3}),$$
$$\ell(\bar{2}|1) - \ell(1|2) = 2 * 1 + (2n-1) * 1 - 1, \text{ and } \ell(\bar{2}|\bar{3}) - \ell(\bar{2}|1) = 2 * 0 + (2n-1) * 1 - 1.$$

Next, there is clearly a path from $(a|b)$ to $(1|2)$ by decomposing the permutation $(a|b)$ into simple reflections. Finally we claim that there is a path from $(\bar{2}|\bar{3})$ to any other vertex. Indeed, if $(a|b) \in W^{odd} \backslash \{(\bar{2}|\bar{3})\}$ then there is another point $(c|d) \in W^{odd}$ and edge $(c|d) \to (a|b)$ such that $\ell(c|d) - \ell(a|b) = 1$. Since the $\ell(\bar{2}|\bar{3})$ is maximum in the Bruhat order, we conclude the combinatorial quantum Bruhat graph is strongly connected.

The quantum Bruhat graph $\mathcal{G}_n$ has a cycle of length 2 given by

$$(1|2) \to (2|1) \to (1|2).$$

There is a cycle of length $2n-1$ given by one of the following two cases. For $n = 2$ we have

$$(1|2) \to (1|\bar{3}) \to (1|3) \to (1|2).$$

For $n > 2$ we have

$$(1|2) \to (1|\bar{3}) \to \cdots \to (1|\bar{n}) \to (1|n) \to \cdots \to (1|2).$$

The result follows. $\square$

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