MINIMUM QUANTUM DEGREES WITH MAYA DIAGRAMS

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Abstract. We use Maya diagrams to refine the criterion by Fulton and Woodward for the smallest powers of the quantum parameter q that occur in a product of Schubert classes in the (small) quantum cohomology of partial flags. Our approach using Maya diagrams yields a combinatorial proof that the minimal quantum degrees are unique for partial flags. Furthermore, visual combinatorial rules are given to perform precise calculations.

1. Introduction

Let $I = \{i_0 := 0 < i_1 < i_2 < \cdots < i_k < i_{k+1} := n\}$. Let $\mathrm{Fl} := \mathrm{Fl}(I;n)$ denote the partial flag given by

$$\mathrm{Fl}(I;n) := \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n : \dim V_j = i_j\}.$$

Let $\mathrm{QH}^*(\mathrm{Fl})$ be the small quantum cohomology with Schubert classes σ_w, $w \in W^P$. The set W^P is the minimum length coset representatives of the associated Weyl group W moded out by a parabolic P that corresponds to the set I. The set W^P is defined in Section 2. We denote the Poincare dual of σ_v by σ^v or $\sigma_{v^\vee}$. The small quantum cohomology ring $\mathrm{QH}^*(\mathrm{Fl})$ is a graded $\mathbb{Z}[q]$-module. The multiplication is given by

$$\sigma^v \star \sigma_w = \sum_{u,d \geqslant 0} c_{v^\vee,w}^{u,d} q^d \sigma_u$$

where $c_{v^\vee,w}^{u,d}$ is the Gromov-Witten invariant that enumerates the degree d rational curves. Given any element $\tau \in \mathrm{QH}^*(\mathrm{Fl})$, we say that q^d **occurs** in τ if the coefficient of $q^d \sigma_w$ is not zero for some $w \in W^P$.

The purpose of the article is to use Maya diagrams to refine a criterion by Fulton and Woodward in [FW04] for the smallest powers of the quantum parameter q that occur in a product of Schubert classes in the (small) quantum cohomology of partial flag. Using the moment graph this requires many cases to be checked to find the minimum quantum degree. Maya diagrams reduce this process down to a single calculation. The Maya diagram approach can be thought of as a generalization of removing rim hooks on Young Tableau in the Grassmannian case that is presented in [BCFF99]. The results in this article are combinatorial in the sense that we use Maya diagrams to describe the chains in the moment graph that Fulton and Woodward defined in [FW04] (see Proposition 2.2).

Furthermore, our approach using Maya diagrams yields a combinatorial proof that the minimal quantum degrees are in fact unique for partial flags. With geometric techniques, Postnikov proved that the minimum quantum degree is unique for G/B in [Pos05b, Corollary 3]. This result was later extended to general homogeneous space G/P in [BCLM20] also using geometric techniques. Minimum quantum degrees are also studied in [Pos05a, Buc03, Yon03, Bel04, Bä22, SW20].

Maya diagrams for partial flags give a characterization of the Bruhat order by slightly modifying a theorem by Proctor in [Pro82, Theorem 5A] and stated herein as Proposition

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3.7. Furthermore, the notion of generalized rim hooks is defined in Definition 4.3. With these two key notions, there is a canonical lower bound for the minimal quantum degrees as stated in Lemma 4.18. We then show that this lower bound is achieved in Theorem 6.1. Next we begin with preliminaries to state and prove our main results. We expect analogous results to hold for types B, C, and D. Since the Weyl groups are different for each type, the definition of the Maya diagrams will need to be modified. Next we begin with preliminaries to state and prove our main results.

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2. Preliminaries

Let $I = \{i_0 := 0 < i_1 < i_2 < \cdots < i_k < i_{k+1} := n\}$. Let $\mathrm{Fl} := \mathrm{Fl}(I;n)$ denote the partial flag given by

$$\mathrm{Fl}(I;n) := \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n : \dim V_j = i_j\}.$$

Consider the root system of type A_{n-1} with positive roots $R^+ = \{e_l - e_m : 1 \leqslant l < m \leqslant n\}$ and the subset of simple roots $\Delta = \{\alpha_l := e_l - e_{l+1} : 1 \leqslant l \leqslant n-1\}$. The associated Weyl group W is S_n. For $1 \leqslant l \leqslant n-1$ denote by s_l the simple reflection corresponding to the root $e_l - e_{l+1}$. Each $I = \{i_0 := 0 < i_1 < i_2 < \cdots < i_k < i_{k+1} := n\}$ determines a parabolic subgroup P_I with Weyl group $W_{P_I} = \langle s_l : l \neq i_j \rangle$ generated by reflections with indices not in I. We will define $P := P_I$ for notational ease. Let $\Delta_P := \{\alpha_{i_s} : i_s \notin \{i_1, \cdots, i_k\}\}$ and $R_P^+ := \mathrm{Span}\Delta_P \cap R^+$; these are the positive roots of P. Let $\alpha \in R^+ \backslash R_P^+$. Then $\alpha + \Delta_P$ is the sum of simple roots in $R^+ \backslash R_P^+$ given by

$$\alpha + \Delta_P = \sum_{j=1}^{k} d_j(e_{i_j} - e_{i_{j+1}}) + \Delta_P$$

For notational ease, we will denote this sum by the k-tuple $(d_1, d_2, \cdots, d_k)$. Let $\ell : W \to \mathbb{N}$ be the length function and denote by W^P the set of minimal length representatives of the cosets of W/W_P. The length function descends to W/W_P by $\ell(uW_P) = \ell(u')$ where $u' \in w^P$ is the minimal length representative for the coset uW_P. We have a natural ordering $1 < 2 < \cdots < n$. Since $w(i_k + 1) < \cdots < w(n)$ are determined then we will identify the elements of W^P with

$$(w(1) < \cdots < w(i_1) | w(i_1 + 1) < \cdots < w(i_2) | \cdots | w(i_{k-1} + 1) < \cdots < w(i_k)).$$

Furthermore, there are times in the paper where we need to consider coset representatives not in W^P, in those instances we write

$$(w(1), \cdots, w(i_1) | w(i_1 + 1), \cdots, w(i_2) | \cdots | w(i_{k-1} + 1), \cdots, w(i_k))$$

where the entries between the vertical bars may be interchanged.

2.1. **Chains.** An edge in the moment graph corresponds to a torus stable curve of a fixed degree. A chain along those edges corresponds to a torus stable curve where the degree is the sum of the edge degrees in the chain. So, studying chains in the moment graph gives us information about curves in the flag variety. Here we will follow the exposition of [FW04] and specialize to the case of partial flags. We say that two unequal elements v and w in W^P are **adjacent** if there is a reflection $s_{e_l - e_m} \in W$ such that $w = v s_{e_l - e_m}$. The reflection $e_l - e_m$ the sum of simple reflection in $R^+ \backslash R_P^+$. That is, if $i_{a-1}+1 \leqslant l \leqslant i_a$ and $i_{b-1}+1 \leqslant m \leqslant i_b$ then define

$$\begin{aligned} d(v,w) &= e_l - e_m + \Delta_P = (e_{i_a} - e_{i_a+1}) + \cdots + (e_{i_{b-1}} - e_{i_{b-1}+1}) + \Delta_P \\ &= (\overbrace{0,\cdots,0}^{a-1},\overbrace{1,\cdots,1}^{b-a},\overbrace{0,\cdots,0}^{k+1-b}). \end{aligned}$$

Define a **chain** $\mathcal{C}$ from v to w in W^P to be a sequence $u_0, u_1, \cdots, u_r$ in W^P such that u_{i-1} and u_i are adjacent for $1 \leqslant i \leqslant r$ and $u_0 \leqslant v$ and $w \leqslant u_r$. We say the chain **originates** at u_0 and **terminates** at u_r. For any chain $u_0, u_1, \cdots, u_r$ we define the **degree** of the chain $\mathcal{C}$, denoted $\deg_{\mathcal{C}}(v,w)$, to be the sum of the degrees $d(u_{i-1}, u_i)$ for $1 \leqslant i \leqslant r$. Note that there is a chain of degree 0 between v and w exactly when $w \leqslant v$.

2.2. **Quantum Cohomology.** Let $\mathrm{QH}^*(\mathrm{Fl})$ denote the quantum cohomology ring of Fl. The Schubert classes σ_w, $w \in W^P$, form a basis. Let $\sigma^w := \sigma_w^\vee$ be the Poincare dual of σ_w for any $w \in W^P$. Take a variable q_j for each $i_j \in I$ with $1 \leqslant j \leqslant k$, and let $\mathbb{Z}[q]$ be the polynomial ring with these q_j as indeterminates where $\deg q_j = i_{j+1} - i_{j-1}$. For a degree $d = (d_1, \cdots, d_k)$ that corresponds to $\sum_{j=1}^k d_j \sigma^{s_{i_j}} \in H_2(\mathrm{Fl})$ (this is an integral sum of curve classes), we write $q^d = \Pi_{j=1}^k q_j^{d_j}$. The small quantum cohomology ring $\mathrm{QH}^*(\mathrm{Fl})$ is a graded $\mathbb{Z}[q]$-module. The multiplication is given by

$$\sigma^v \star \sigma_w = \sum_{u, d \geqslant 0} c_{v^\vee, w}^{u,d} q^d \sigma_u$$

where $c_{v^\vee,w}^{u,d}$ is the Gromov-Witten invariant that enumerates the degree d rational curves. See [Buc05] for details.

Remark 2.1. The q indeterminates would be $q_1, \cdots, q_k$ with the degree coming from I. So, different subsets of I with k elements correspond to the same set of indeterminates but different grading.

2.3. **Hecke Product.** The purpose of using the Hecke product comes from the work of Buch and Mihalcea in [BM15]. In particular, they use the Heck product to calculate curve neighborhoods of Schubert varieties which are the closures of degree d rational curves that intersect a given Schubert variety. The curve neighborhood behavior is intimately related to minimum quantum degrees which motives the use of Heck products in this manuscript

The Weyl group W admits a partial ordering $\leqslant$ given by the **Bruhat order**. Its covering relations are given by $w < ws_\alpha$ where $\alpha \in R^+$ is a root and $\ell(w) < \ell(ws_\alpha)$. We will use the **Hecke product** on the Weyl group W. For a simple reflection s_i the product is defined by

$$w \cdot s_i = \begin{cases} ws_i & \text{if } \ell(ws_i) > \ell(w); \\ w & \text{otherwise.} \end{cases}$$

If $v = s_{i_1} s_{i_2} \cdots \cdot s_{i_t}$ then $w \cdot v = w \cdot s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_t}$. It is shown in [BM15, Section 3] that this product is independent of the chosen reduced expression for v. The Hecke product gives W a structure of an associative monoid; see e.g. [BM15, §3] for more details. For any parabolic group P, the Hecke product determines a left action $W \times W/W_P \to W/W_P$ defined by

$$u \cdot (wW_P) = (u \cdot w) W_P.$$

See the paragraph following [BM15, Proposition 3.2].

2.4. **Fulton and Woodward's formula for minimal quantum degrees.** Given any element $\tau \in \mathrm{QH}^*(\mathrm{Fl})$, we say that q^d **occurs** in τ if the coefficient of $q^d\sigma_w$ is not zero for some w. The following result provides an equivalent definition to degrees in terms of chains the Bruhat graph.

Proposition 2.2. [FW04, Theorem 9.1] *Let $v, w \in W^P$, and let d be a degree. The following are equivalent:*

(1) There is a degree $c \leqslant d$ such that q^c occurs in $\sigma^v \star \sigma_w$.
(2) There is a chain of degree $c \leqslant d$ between v and w.

3. Maya Diagrams

In this section, we give the definition of Maya diagrams. Maya diagrams show up in different contexts. (see e.g. [DJK+89, LR19, CGUGM2001]). We will also describe the Bruhat order in terms of Maya diagrams.

Definition 3.1. Let $w \in W^P$. The Maya diagram M^w corresponding to w is an $(k+1)\times n$ grid with the southwest corner chosen to be $(1,1)$ box and we index with $(\mathrm{rows}, \mathrm{columns})$. We place an x in the $(y, w(i))$ position for $1 \leqslant j \leqslant k+1$, $1 \leqslant i \leqslant i_j$, and $j \leqslant y \leqslant k+1$. We color the bottom x of each column black with all other x's blue. We denote the row indexed by y as m^w_y.

Each row corresponds to an increasing interval in the permutation, and you can read out the one-line notation by following the black x's row by row from the bottom.

Example 3.2. The minimal length representatives $w = (1|5 < 9|10 < 11|4 < 6|2 < 7)$ and $v = (2|7 < 11|10 < 12|8 < 9|1 < 5)$ corresponds to the Maya diagrams

$M^w =$

x	x	x	x	x	x	x	x	x	x	x	x
x	x		x	x	x	x		x	x	x	
x			x	x	x			x	x	x	
x				x				x	x	x	
x				x				x			
x											

and $M^v =$

x	x	x	x	x	x	x	x	x	x	x	x
x	x			x		x	x	x	x	x	x
	x					x	x	x	x	x	x
	x					x			x	x	x
	x					x				x	
	x										

.

In M^w, a black x is placed in the bottom row and the 1st column; two black x's are placed in the 2nd row up in the 5th and 9th columns. This corresponds to the $1, 5$ and 9 in $w = (1|5 < 9|10 < 11|4 < 6|2 < 7)$.

Let M^w be the Maya diagram corresponding to $w \in W^P$ and let $1 \leqslant y \leqslant k$. Let $\pi_y : W^P \to W^{P_{i_y}}$ denote the natural projection. Then $M^{\pi_y(w)}$ is a Maya diagram with two rows and n columns with the top row having an x in each position and the bottom row is m^w_y.

Example 3.3.

$M^w =$

x	x	x	x	x	x	x	x	x	x	x	x
x	x		x	x	x	x		x	x	x	
x			x	x	x			x	x	x	
x				x				x	x	x	
x				x				x			
x											

and $M^{\pi_3(w)} =$

x	x	x	x	x	x	x	x	x	x	x	x
x				x				x	x	x	

.

3.1. **Bruhat order with Maya diagrams.** We begin the subsection with technical definitions.

Definition 3.4. Let $w, v \in W^P$. Let M^w be the Maya diagram that corresponds to $w \in W^P$.

(1) Define

$$f(M^w, y, z) := \begin{cases} x & y = j, z = w(i), \text{ for some } i, j \text{ with } 1 \leqslant i \leqslant i_j \text{ and } 1 \leqslant j \leqslant k+1; \\ 0 & \text{otherwise} \end{cases}.$$

(2) Define $S_y(M^w, z) := \#\{i : f(M^w, y, i) = x \text{ for } 1 \leqslant i \leqslant z\}$.

(3) We say that $M^w \leqslant M^v$ if $S_y(M^w, z) \geqslant S_y(M^v, z)$ for all y and z such that $1 \leqslant z \leqslant n$ and $1 \leqslant y \leqslant k+1$.

Example 3.5. Consider $w = (1|5 < 9|10 < 11|4 < 6|2 < 7)$. Then we have

$$M^w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & & x & x & x & x & & x & x & x & \\ \hline x & & & x & x & x & & & x & x & x & \\ \hline x & & & & x & & & & x & x & x & \\ \hline x & & & & x & & & & x & & & \\ \hline x & & & & & & & & & & & \\ \hline \end{array}.$$

We are considering the conditions where $y = j, z = w(i)$, for some i, j with $1 \leqslant i \leqslant i_j$ and $1 \leqslant j \leqslant k+1$. Here, $i_1 = 1$, $i_2 = 3$ and $w(2) = 5$. This means the fifth column is marked with an x in the second row since $1 \leqslant 2 \leqslant i_2$ but $2 > i_1$. Since $1 \leqslant 2 \leqslant i_j$ for $2 \leqslant j \leqslant k+1$, there is an x in the fifth column and jth row for $2 \leqslant j \leqslant k+1$. In particular, have that $f(M^w, 3, 5) = x$ which corresponds to the x in M^w.

Example 3.6. Recall the Maya diagrams from Example 3.2.

$$M^w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & & x & x & x & x & & x & x & x & \\ \hline x & & & x & x & x & & & x & x & x & \\ \hline x & & & & x & & & & x & x & x & \\ \hline x & & & & x & & & & x & & & \\ \hline x & & & & & & & & & & & \\ \hline \end{array} \text{ and } M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & & & x & & x & x & x & x & x & x \\ \hline & x & & & & & x & x & x & x & x & x \\ \hline & x & & & & & x & & & x & x & x \\ \hline & x & & & & & x & & & & x & \\ \hline & x & & & & & & & & & & \\ \hline \end{array}$$

Next we have a few examples of computations color coordinated to match with the x that are being counted in the Maya diagrams.

$$S_1(M^w, 1) = 1 \geqslant 0 = S_1(M^v, 1) \text{ and } S_3(M^w, 9) = 3 \geqslant 2 = S_3(M^v, 9).$$

In this example, we have that $M^w \leqslant M^v$.

Next we present a proposition that relates the Bruhat order on W^P with the partial order on Maya Diagrams. This is another way of presenting the result in [Pro82, Theorem 5A].

Proposition 3.7. [Pro82, Theorem 5A] *Let $w, v \in W^P$. Then $w \leqslant v$ if and only if $M^w \leqslant M^v$.*

4. Maya diagram combinatorics

In the section we will give the definition of the generalized rim hook rule and study chains in terms of Maya diagrams.

4.1. **Generalized rim hook rule.** Maya diagrams give a way to see a generalized rim hook rule. The generalize (a,b)-rim hook connects the combinatorics of the Maya Diagram and Heck product to curves of degree

$$(\overbrace{0,\cdots,0}^{a-1},\overbrace{1,\cdots,1}^{b-a},\overbrace{0,\cdots,0}^{k+1-b}).$$

This can be thought of as a generalization of removing rim hooks on Young Tableau in the Grassmannian case that is presented in [BCFF99]. We now define the generalized rim hook rule.

Definition 4.1. Let $v \in W^P$ and let M^v be the corresponding Maya diagram. For $1 \leqslant y \leqslant k$ define

$$\phi(M^v,y) := \min\{z : f(M^v,y,z) = x \text{ and } f(M^v,y-1,z) = 0\},$$
$$\psi(M^v,y) := \max\{z : f(M^v,y+1,z) = x \text{ and } f(M^v,y,z) = 0\}.$$

Example 4.2. Consider the following Maya diagram.

$$M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & & & x & & x & x & x & x & x & x \\ \hline & x & & & & & x & x & x & x & x & x \\ \hline & x & & & & & x & & & x & x & x \\ \hline & x & & & & & x & & & & x & \\ \hline & x & & & & & & & & & & \\ \hline \end{array}.$$

Here $\phi(M^v,3) = 10$ and $\psi(M^v,4) = 5$ where the colors correspond to the x in the definitions of ϕ and ψ.

Next we define the generalized rim hook rule.

Definition 4.3. Let a,b be such that $1 \leqslant a < b \leqslant k+1$, and let M^v be a Maya diagram for $v \in W^P$. We define the *generalized* (a,b)*-rim hook* as the Maya diagram obtained by the following process:

(1) Let $M^v_{\uparrow a} := M^v$.
(2) For $a \leqslant j \leqslant b-1$, define $M^v_{\uparrow j+1}$ from $M^v_{\uparrow j}$ by removing the x in position $(j,\phi(M^v_{\uparrow j},j))$.
(3) Let $M^v_{\downarrow b} := M^v_{\uparrow b}$.
(4) For $b-1 \geqslant j \geqslant a$, define $M^v_{\downarrow j}$ from $M^v_{\downarrow j+1}$ by adding an x to position $(j,\psi(M^v_{\downarrow j},j))$.
(5) The completed generalized (a,b)-rim hook is given by $M^v_{\downarrow a}$.

We refer to this process as the (a,b)*-rim hook rule.*

Definition 4.3 is well-defined since we are assuring the number of x's in each row is the same at the beginning and the end of the process and that any x, except those in the top row, has an x above it.

Remark 4.4. Definition 4.3 corresponds to the rim hook for Grassmannians (i.e. the $k=1$ case). See [FW04].

Example 4.5. This example is in the Grassmannian $\mathrm{Fl}(\{0<8<12\};12)$ with

$$v = (1<2<3<5<8<9<11<12) \in W^P.$$

Here we will calculate the generalized $(1,2)$-rim hook. The symbols $\uparrow$ and $\downarrow$ describe the movement of the x's.

$$M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & x & & & x & x & & x & x \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline x & x & x & x & x & x & x & x & x & \downarrow & x & x \\ \hline \uparrow & x & x & & x & & & x & x & x & x & x \\ \hline \end{array}$$

$=$

x	x	x	x	x	x	x	x	x	x	x	x
	x	x		x			x	x	x	x	x

.

Example 4.6. The following is an example of a generalized $(2, 6)$-rim hook in

$$\mathrm{Fl}(\{0 < 1 < 3 < 5 < 7 < 9 < 12\}; 12).$$

Here $v = (2|3 < 8|10 < 12|9 < 11|1 < 5) \in W^P$.

$M^v =$

x	x	x	x	x	x	x	x	x	x	x	x
x	x	x		x			x	x	x	x	x
	x	x					x	x	x	x	x
	x	x					x		x		x
	x	x					x				
	x										

$\longrightarrow$

x	x	x	x	x	x	$\downarrow$	x	x	x	x	x
$\uparrow$	x	x		x		$\downarrow$	x	x	x	x	x
	x	$\uparrow$				x	x	x	x	$\downarrow$	x
	x	$\uparrow$					x		x	x	$\downarrow$
	x	$\uparrow$					x				x
	x										

$=$

x	x	x	x	x	x	x	x	x	x	x	x
	x	x		x		x	x	x	x	x	x
	x					x	x	x	x	x	x
	x						x		x	x	x
	x						x				x
	x										

.

4.2. **Chains in terms of Maya diagrams.** The first lemma connects the generalized (a, b)-rim hook rule to the Hecke product and constructs a chain. We will now state a technical definition before describing Chains in terms of Maya diagrams.

Definition 4.7. We say that two postive roots $e_a - e_b, e_c - e_d \in R^+$ intersect at most at an end point if $a < b \leqslant c < d$ or $c < d \leqslant a < b$.

Definition 4.8. Let $\{\lambda_q\}_{q=1}^{q}, \{\beta_j\}_{j=1}^{J} \subset R^+$. We say that $\sum \lambda_q \geqslant \sum \beta_j$ if $\sum \lambda_q - \sum \beta_j$ is a nonnegative linear combination of positive roots.

Lemma 4.9. *Let M^v be a Maya diagram that corresponds to $v \in W^P$. Apply the (a, b)-rim hook rule to M^v and call the resulting Maya diagram $M^{v'}$ where $v \in W^P$. Then we have the following:*

(1) $s_{e_{i_{a-1}+1}-e_{i_b}} = s_{i_{a-1}+1}s_{i_{a-1}+2}\cdots s_{i_b-2}s_{i_b-1}s_{i_b-2}\cdots s_{i_{a-1}+2}s_{i_{a-1}+1}$.

(2) $v' = v \cdot s_{e_{i_{a-1}+1}-e_{i_b}}$.

(3) There exists a sequence of positive roots $\{\beta_j\}_{j=1}^{J} \subset R^+$ such that

- *(a)* $e_{i_{a-1}+1} - e_{i_b} \geqslant \sum \beta_j$;
- *(b) any two elements of $\{\beta_j\}_{j=1}^{J}$ overlap at most at an end point;*
- *(c)* $v' = v s_{\beta_1} \ldots s_{\beta_J}$.

Proof. We will prove each case individually.

(1) This is the result of a direct computation.

(2) First, without loss of generality by re-indexing the word from Part (1), we only need to consider the case $s_{e_1-e_{i_n}} = s_1 s_2 \ldots s_{n-2}s_{n-1}s_{n-2}\ldots s_2 s_1$. If they exist, find the smallest indices $j_1, 1 \leqslant j_1 \leqslant n-1$ and $j_1', 1 \leqslant j_1' < n-1$ such that

(a) $\ell(v \cdot s_1 s_2 \ldots s_{j_1}) < \ell(v \cdot s_1 s_2 \ldots s_{j_1-1})$ and

(b) $\ell(v \cdot s_1 s_2 \ldots s_{n-2}s_{n-1}s_{n-2}\ldots s_{j_1'}) < \ell(v \cdot s_1 s_2 \ldots s_{n-2}s_{n-1}s_{n-2}\ldots s_{j_1'-1})$.

Case 1: Suppose that j_1 exists but j_1' does not. If $j_1 = n-1$ then $v(n) = 1$. But this implies $\ell(v \cdot s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2}) < \ell(v \cdot s_1 s_2 \dots s_{n-2} s_{n-1})$ which implies j_1' exists. So, $1 \leqslant j_1 < n-1$. Then we have the following

$$\begin{aligned} v \cdot s_{e_1 - e_n} &= v \cdot s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1 \\ &= v \cdot s_1 s_2 \dots s_{j_1 - 1} s_{j_1 + 1} \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1 \\ &= (v \cdot (s_{j_1+1} \dots s_{n-2} s_{n-1} s_{n-2} \dots s_{j_1+1}))\, (s_1 s_2 \dots s_{j_1-1} s_{j_1} s_{j_1-1} s_2 s_1) \\ &= \left(v \cdot s_{e_{j_1+1} - e_n}\right) \left(s_{e_1 - e_{j_1+1}}\right). \end{aligned}$$

Case 2: Suppose that j_1 does not exist but j_1' does. Then we have the following

$$\begin{aligned} v \cdot s_{e_1 - e_n} &= v \cdot s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1 \\ &= v \cdot s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_{j_1'+1} s_{j_1'-1} \dots s_2 s_1 \\ &= \left(v \left(s_1 s_2 \dots s_{j_1'-1} s_{j_1'} s_{j_1'-1} \dots s_2 s_1\right)\right) \cdot (s_{j_1+1} \dots s_{n-2} s_{n-1} s_{n-2} \dots s_{j_1+1}) \\ &= \left(v \left(s_{e_1 - e_{j_1'+1}}\right)\right) \cdot \left(s_{e_{j_1+1} - e_n}\right). \end{aligned}$$

Case 3: Suppose that j_1 and j_1' both exist, $j_1 > j_1'$, and $j_1 \neq n-1$. Then we have the following

$$\begin{aligned} & v \cdot s_{e_1 - e_n} \\ =\ & v \cdot s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1 \\ =\ & v \cdot s_1 s_2 \dots s_{j_1-1} s_{j_1+1} \dots s_{n-2} s_{n-1} s_{n-2} \dots s_{j_1'+1} s_{j_1'-1} \dots s_2 s_1 \\ =\ & \left(v s_1 s_2 \dots s_{j_1'-1} s_{j_1'} s_{j_1'-1} \dots s_2 s_1\right) \cdot \left(s_{j_1'+1} s_{j_1'+2} \dots s_{j_1-1} s_{j_1+1} \dots s_{n-2} s_{n-1} s_{n-2} \dots s_{j_1'+1}\right) \\ =\ & \left(\left(v s_1 s_2 \dots s_{j_1'-1} s_{j_1'} s_{j_1'-1} \dots s_2 s_1\right) \cdot (s_{j_1+1} \dots s_{n-2} s_{n-1} s_{n-2} \dots s_{j_1+1})\right) \left(s_{j_1'+1} s_{j_1'+2} \dots s_{j_1-1} s_{j_1} s_{j_1-1} \dots s_{j_1'+2} s_{j_1'+1}\right) \\ =\ & \left(\left(v s_{e_1 - e_{j_1'}}\right) \cdot \left(s_{e_{j_1+1} - e_n}\right)\right) \cdot \left(s_{e_{j_1'+1} - e_{j_1+1}}\right). \end{aligned}$$

Case 4: Suppose that j_1 and j_1' both exist, $j_1 > j_1'$, and $j_1 = n-1$. Then we have the following

$$v \cdot s_{e_1 - e_n} = \left(v s_{e_1 - e_{j_1'}}\right) \left(s_{e_{j_1'+1} - e_n}\right).$$

Case 5: Suppose that j_1 and j_1' both exist, $j_1 < j_1'$. Then we have the following

$$v \cdot s_{e_1 - e_n} = \left(\left(v \cdot s_{e_{j_1+1} - e_{j_1'+1}}\right) \cdot \left(s_{e_{j_1'+1} - e_n}\right)\right) \left(s_{e_1 - e_{j_1}}\right).$$

Case 6: Suppose that j_1 and j_1' both exist and $j_1 = j_1'$. Then we have the following

$$v \cdot s_{e_1 - e_n} = v s_{e_{j_1+1} - e_n}.$$

A key observation is that the (non-Hecke) permutation multiplication of v by $s_{e_1 - e_{j_1}}$ or $s_{e_1 - e_{j_1'}}$ appears. Therefore, after iterating using the 6 cases above, the Hecke product $v \cdot s_{e_1 - e_n}$ may be written as a (non-Hecke) product of v times reflection of the form $s_{e_1 - e_{j_1}}$ or $s_{e_1 - e_{j_1'}}$ re-indexed as appropriate. When chained together, these reflections correspond to $M^v_{\uparrow j}$ and $M^v_{\downarrow j}$ from Definition 4.3. Therefore, $v' = v \cdot s_{e_{i_{a-1}+1} - e_{i_b}}$.

(3) This is an immediate consequence of applying iterations of the 6 cases in Part (2). The result follows.

$\square$

Example 4.10. Consider the partial flag variety $\mathrm{Fl}(\{0 < 1 < 3 < 5 < 7 < 9 < 12\}, 12)$. Let $v = (2|3 < 8|10 < 12|9 < 11|1 < 5) \in W_P$. Here we will show the application of a $(2,6)-$rim hook to v. The outcome is $v \cdot s_{e_2 - e_{12}}$ by Part (1) of Lemma 4.9. The following is an example of Part (3) of Lemma 4.9 and the procedure to produce the sequence reflections.

(1) For the first step we have the following two facts.
  (a) The number 12 is the biggest number in entries 2 through 12 (recall entries 9 through 12 are suppressed).
  (b) The number 3 is the smallest number in an entry before 12 and in entries 2 through 12.

So we will use the reflection $s_{e_2-e_5}$ to interchange 3 and 12 to find

$$(2|3, 8|10, 12|9, 11|1, 5)s_{e_2-e_5} = (2|12, 8|10, 3|9, 11|1, 5).$$

This is viewed visually in terms of Maya diagrams. Here we start by labeling the black x's with their initial position reading bottom to top and left to right. In rows 2 through 6, notice that the x (labeled 5) farthest to the right is in the 12th column and the x (labeled 2) farthest to the left with a label less than 5 is in the third column. So, we interchange columns 3 and 12 using $s_{e_2-e_5}$. The entries $x2, x3, x4$ are ineligible to be considered in future steps.

x	x	x	$x10$	x	$x11$	$x12$	x	x	x	x	x
$x8$	x	x		$x9$			x	x	x	x	x
	x	x					x	$x6$	x	$x7$	x
	x	x					x		$x4$		$x5$
	x	$x2$					$x3$				
	$x1$										

$\xrightarrow{s_{e_2-e_5}}$

x	x	x	$x10$	x	$x11$	$x12$	x	x	x	x	x
$x8$	x	x		$x9$			x	x	x	x	x
	x	x					x	$x6$	x	$x7$	x
	x	$x5$					x		$x4$		x
	x						$x3$				$x2$
	$x1$										

.

(2) For the second step we have the following two facts.
  (a) The number 11 is the biggest number in entries 5 through 12.
  (b) The number 3 is the smallest number in an entry before 12 and in entries 5 through 12.

So we will use the reflection $s_{e_5-e_7}$ to interchange 3 and 11 to find

$$(2|12, 8|10, 3|9, 11|1, 5)s_{e_5-e_7} = (2|12, 8|10, 11|9, 3|1, 5).$$

This is viewed visually in terms of Maya diagrams. In rows 2 through 6, notice that the x (labeled 7) farthest to the right is in the 11th column and the x (labeled 5) farthest to the left with a label less than 7 is in the 3rd column. So, we interchange columns 3 and 11 using $s_{e_5-e_7}$. The entries $x2, x3, x4, x5, x6$ are ineligible to be considered in future steps.

x	x	x	$x10$	x	$x11$	$x12$	x	x	x	x	x
$x8$	x	x		$x9$			x	x	x	x	x
	x	x					x	$x6$	x	$x7$	x
	x	$x5$					x		$x4$		x
	x						$x3$				$x2$
	$x1$										

$\xrightarrow{s_{e_5-e_7}}$

x	x	x	$x10$	x	$x11$	$x12$	x	x	x	x	x
$x8$	x	x		$x9$			x	x	x	x	x
	x	$x7$					x	$x6$	x	x	x
	x						x		$x4$	$x5$	x
	x						$x3$				$x2$
	$x1$										

.

(3) For the third step we have the following two facts.
  (a) The number 7 is the biggest number in entries 7 through 12.
  (b) The number 1 is the smallest number in an entry before 7 and in entries 7 through 12.

So we will use the reflection $s_{e_8-e_{12}}$ to interchange 1 and 7 to find

$$(2|12,8|10,11|9,3|1,5)s_{e_8-e_{12}} = (2|12,8|10,11|9,3|7,5).$$

This is viewed visually in terms of Maya diagrams. In rows 2 through 6, notice that the x (labeled 12) farthest to the right is in the 7th column and the x (labeled 8) farthest to the left with a label less than 12 is in the 1st column. So, we interchange columns 1 and 7 using $s_{e_8-e_{12}}$. The entries $x2, x3, x4, x5, x6, x9, x10, x11, x12$ are ineligible to be considered in future steps.

x	x	x	$x10$	x	$x11$	$x12$	x	x	x	x	x
$x8$	x	x		$x9$			x	x	x	x	x
	x	$x7$					x	$x6$	x	x	x
	x						x		$x4$	$x5$	x
	x						$x3$				$x2$
	$x1$										

$\xrightarrow{s_{e_8-e_{12}}}$

$x12$	x	x	$x10$	x	$x11$	x	x	x	x	x	x
	x	x		$x9$		$x8$	x	x	x	x	x
	x	$x7$					x	$x6$	x	x	x
	x						x		$x4$	$x5$	x
	x						$x3$				$x2$
	$x1$										

.

(4) For the fourth step we have the following two facts.
  (a) The number 7 is the biggest number in entries 7 through 8.
  (b) The number 3 is the smallest number in an entry before 7 and in entries 7 through 8.

So we will use the reflection $s_{e_7-e_8}$ to interchange 3 and 7 to find

$$(2|12,8|10,11|9,3|7,5)s_{e_7-e_8} = (2|12,8|10,11|9,7|3,5).$$

This is viewed visually in terms of Maya diagrams. In rows 2 through 6, notice that the x (labeled 8) farthest to the right is in the 7th column and the x (labeled 7) farthest to the left with a label less than 8 is in the 3rd column. So, we interchange columns 3 and 7 using $s_{e_7-e_8}$. There are no more eligible x to consider in rows 2 through 6.

$x12$	x	x	$x10$	x	$x11$	x	x	x	x	x	x
	x	x		$x9$		$x8$	x	x	x	x	x
	x	$x7$					x	$x6$	x	x	x
	x						x		$x4$	$x5$	x
	x						$x3$				$x2$
	$x1$										

$\xrightarrow{s_{e_7-e_8}}$

$x12$	x	x	$x10$	x	$x11$	x	x	x	x	x	x
	x	$x8$		$x9$		x	x	x	x	x	x
	x					$x7$	x	$x6$	x	x	x
	x						x		$x4$	$x5$	x
	x						$x3$				$x2$
	$x1$										

.

Finally observe that $v \cdot s_{e_2-e_{12}} = vs_{e_2-e_5}s_{e_5-e_7}s_{e_8-e_{12}}s_{e_7-e_8}$ and $e_2 - e_{12} = (e_2 - e_5) + (e_5 - e_7) + (e_8 - e_{12}) + (e_7 - e_8)$.

Lemma 4.11. *Let M^v be a Maya diagram that corresponds to $v \in W^P$. Apply the (a,b)-rim hook rule to M^v and call the resulting Maya diagram $M^{v'}$ where $v \in W^P$. Then there is a chain $\mathcal{C}$ from originating at v to terminating at v' such that*

$$\deg_{\mathcal{C}}(v,v') \leqslant (\overbrace{0,\cdots,0}^{a-1},\overbrace{1,\cdots,1}^{b-a},\overbrace{0,\cdots,0}^{k+1-b}).$$

Proof. This immediately follows from Part (3) of Lemma 4.9. ☐

The next two definitions give conditions for when to apply generalized (a,b)-rim hooks to produce a chain of minimum degree.

Definition 4.12. Let $v, w \in W^P$ with corresponding Maya diagrams M^v and M^w, respectively. We say position (y,z) in M^v is Bruhat order incompatible with position (y,z)

in M^w if $S_y(M^v, z) > S_y(M^w, z)$. Similarly, we say that row m_y^v in M^v is Bruhat order incompatible with row m_y^w in M^w if $S_y(M^v, z) > S_y(M^w, z)$ for some $1 \leqslant z \leqslant n$. If $S_y(M^v, z) \leqslant S_y(M^w, z)$ for all $1 \leqslant z \leqslant n$, then we say $m_y^w \leqslant m_y^v$.

It is important to apply the (a, b)-rim hook to as many rows as possible that are Bruhat incompatible while not applying it to rows that are Bruhat compatible. This is to reduce the number of (a, b)-rim hooks needed so a minimum quantum degree is calculated.

Definition 4.13. Let M^v and M^w be Maya diagrams with $v, w \in W^P$, respectively, with the following properties:

(1) Row m_j^v in M^v is Bruhat order incompatible with row m_j^w in M^w for $a \leqslant j \leqslant b-1$;
(2) If $a > 1$, then $m_{a-1}^w \leqslant m_{a-1}^v$.
(3) $m_b^w \leqslant m_b^v$.

When the above conditions apply, let $\mathcal{C}_{(v,w)}^{(a,b)}$ denote a chain of degree less than or equal to $(\overbrace{0, \cdots, 0}^{a-1}, \overbrace{1, \cdots, 1}^{b-a}, \overbrace{0, \cdots, 0}^{k+1-b})$ originating at v and terminating at v' where $M^{v'}$ is the result of applying an (a, b)-rim hook rule to M^v.

Example 4.14. Consider the following two Maya diagrams.

$$M^w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & x & x & x & & x & x & x & & \\ \hline x & & & & x & x & x & & x & x & x & & \\ \hline x & & & & x & & & & x & x & x & & \\ \hline x & & & & & & & & x & x & & & \\ \hline x & & & & & & & & & & & & \\ \hline \end{array} \quad \text{and } M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & x & & & x & x & x & x & & x \\ \hline & x & x! & & & & & x & x & x & x & & x \\ \hline & x & x! & & & & & x! & & x & & & x \\ \hline & x & x! & & & & & x! & & & & & \\ \hline & x & & & & & & & & & & & \\ \hline \end{array}.$$

Here row m_j^v in M^v is Bruhat order incompatible with row m_j^w in M^w for $2 \leqslant j \leqslant 4$. Also, $m_1^w \leqslant m_1^v$ and $m_5^w \leqslant m_5^v$. We use ! to mark the Bruhat order incompatible positions in M^v.

Definition 4.15. Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Let $v_0 := v$. Let $\mathcal{C}$ be a chain in W^P in terms of Maya diagrams given by

$$\mathcal{C} : M^{v_0} \xrightarrow{\mathcal{C}_{(v_0,w)}^{(a_0,b_0)}} M^{v_1} \xrightarrow{\mathcal{C}_{(v_1,w)}^{(a_1,b_1)}} M^{v_2} \xrightarrow{\mathcal{C}_{(v_2,w)}^{(a_2,b_2)}} \cdots \xrightarrow{\mathcal{C}_{(v_{r-1},w)}^{(a_{r-1},b_{r-1})}} M^{v_r}.$$

Define $\mathrm{Comp}_y\mathcal{C}$ be the yth component of $\displaystyle\sum_{j=0}^{r-1} \deg \mathcal{C}_{(v_j,w)}^{(a_j,b_j)}$.

The next definition is necessary to state a Lemma 4.18 which states a lower bound for the minimal degrees of chains connecting v to w.

Definition 4.16. Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Let $1 \leqslant y \leqslant k$ and let $\pi_y : W^P \to W^{P_{i_y}}$ be the natural projection where P_{i_y} is the maximal parabolic subgroup associated to i_y. Let $v_0 := v$. *Define* $\deg_y(v, w)$ to be the *smallest* integer such that there is a chain in terms Maya diagrams given by

$$\mathcal{C}(y) : M^{\pi_y(v_0)} \xrightarrow{\mathcal{C}_{(\pi_y(v_0),\pi_y(w))}^{(1,2)}} M^{\pi_y(v_1)} \xrightarrow{\mathcal{C}_{(\pi_y(v_1),\pi_y(w))}^{(1,2)}} \cdots \xrightarrow{\mathcal{C}_{(\pi_y(v_{\deg_y(v,w)-1}),\pi_y(w))}^{(1,2)}} M^{\pi_y(v_{\deg_y(v,w)})}$$

has the property $M^{\pi_y(v_j)} \ngeqslant M^{\pi_y(w)}$ for $1 \leqslant j \leqslant \deg_y(v, w) - 1$ and $M^{\pi_y(v_{\deg_y(v,w)})} \geqslant M^{\pi_y(w)}$.

In Definition 4.16, this a chain of two-row May diagrams where the bottom row corresponds to the y-th row of the original Maya diagram. This corresponds to a minimum quantum degree calculation in the Grassmannian case $\mathrm{Fl}(\{0 < i_y < n\}; n)$.

Example 4.17. See Example 4.5 for an Example of a chain in Definition 4.16.

Lemma 4.18. *Let M^v and M^w be Maya diagrams corresponding to $v, w \in W^P$. Let $\mathcal{C}$ be any chain from v to w. Then*

$$(\deg_1(v,w), \cdots, \deg_k(v,w)) \leqslant \deg_{\mathcal{C}}(v,w).$$

Proof. If not, then for some y, $\deg_y(v,w)$ is not smallest integer such that there is a chain in terms Maya diagrams given by

$$\mathcal{C}(y) : M^{\pi_y(v_0)} \xrightarrow{\mathcal{C}^{(1,2)}_{(\pi_y(v_0),\pi_y(w))}} M^{\pi_y(v_1)} \xrightarrow{\mathcal{C}^{(1,2)}_{(\pi_y(v_1),\pi_y(w))}} \cdots \xrightarrow{\mathcal{C}^{(1,2)}_{(\pi_y(v_{\deg_y(v,w)-1}),\pi_y(w))}} M^{\pi_y(v_{\deg_y(v,w)})}$$

has the property $M^{\pi_y(v_j)} \ngeqslant M^{\pi_y(w)}$ for $1 \leqslant j \leqslant \deg_y(v,w) - 1$ and $M^{\pi_y(v_{\deg_y(v,w)})} \geqslant M^{\pi_y(w)}$. □

5. Maya diagrams and the Bruhat order

It is not clear that $\deg_y(v,w)$ and the yth component of $\sum_{j=0}^{r-1} \deg \mathcal{C}^{(a_j,b_j)}_{(v_j,w)}$ are equal. This is because generalized (a,b)-rim hooks do not necessarily remove the first x in a row and place an x in the last open position in a particular row. We use Lemmas 5.1, 5.2, and 5.4 to address this question. The next lemma addresses the Bruhat compatibility of adding an x to a row when applying a generalized (a,b)-rim hook rule.

Lemma 5.1. *Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Consider*

$$M^v \xrightarrow{\mathcal{C}^{(a,b)}_{(v,w)}} M^{v'}.$$

Consider the yth rows where $a \leqslant y \leqslant b-1$. Suppose that the new x in $m_y^{v'}$ that is not in m_y^v is in position (y, z_0). It follows that $S_y(M^{v'}, z) \leqslant S_y(M^w, z)$ for all $z \geqslant z_0$.

Proof. For a contradiction assume that $S_y(M^{v'}, z) > S_y(M^w, z)$ for some $z \geqslant z_0$. Then by the generalized (a,b)-rim hook rule we must have that

$$S_{y+1}(M^{v'}, z) = S_y(M^{v'}, z) + (i_{y+1} - i_y).$$

Also,

$$S_y(M^w, z) + (i_{y+1} - i_y) \geqslant S_{y+1}(M^w, z).$$

It follows that $S_{y+1}(M^{v'}, z) > S_{y+1}(M^w, z)$. Therefore $S_b(M^{v'}, z) > S_b(M^w, z)$ by repeating the previous argument. This is a contradiction since $m_b^{v'} \geqslant m_b^w$. It follows that $S_y(M^{v'}, z) \leqslant S_y(M^w, z)$ for $z \geqslant z_0$. □

The next lemma addresses the Bruhat compatibility of not necessarily removing the first x in a row when applying a generalized (a,b)-rim hook rule.

Lemma 5.2. *Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Consider*

$$M^v \xrightarrow{\mathcal{C}^{(a,b)}_{(v,w)}} M^{v'}.$$

Consider the yth rows where $a \leqslant y \leqslant b-1$. Suppose that the x in m_y^v that is not in $m_y^{v'}$ is in position (y, z_0). It follows that $S_y(M^{v'}, z) \leqslant S_y(M^w, z)$ for all $z < z_0$.

Proof. If $a = 1$ then $f(M^v, y, z) = 0$ for all $z < z_0$. So, $S_1(M^{v'}, z) \leqslant S_1(M^w, z)$ for all $z < z_0$.

Suppose $a > 1$ and $z < z_0$. By the definition of $\mathcal{C}^{(a,b)}_{(v,w)}$, we know that $S_{a-1}(M^v, z) \leqslant S_{a-1}(M^w, z)$. It must be true that $S_y(M^{v'}, z) = S_{a-1}(M^v, z)$ since no x is added or removed from the y-th row that is before (y, z_0) position. Also observe that $S_y(M^w, z)$ increases as y increases. Therefore,

$$S_y(M^{v'}, z) = S_{a-1}(M^v, z) \leqslant S_{a-1}(M^w, z) \leqslant S_y(M^w, z).$$

The result follows. $\square$

Example 5.3. Consider the following two Maya diagrams.

$$M^w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & x & x & x & & x & x & x & & \\ \hline x & & & & x & x & x & & x & x & x & & \\ \hline x & & & & x & & & & x & x & x & & \\ \hline x & & & & & & & & x & x & & & \\ \hline x & & & & & & & & & & & & \\ \hline \end{array} \text{ and } M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & x & & & x & x & x & x & & x \\ \hline & x & x & & & & & x & x & x & x & & x \\ \hline & x & x & & & & & x & & x & & & x \\ \hline & x & x & & & & & x & & & & & \\ \hline & x & & & & & & & & & & & \\ \hline \end{array}.$$

Then we have the following where ! indicate the Bruhat order incompatible positions and $\uparrow$ and $\downarrow$ describe the movement of the x's for when a (2,5)-rim hook is applied. The entries in brown are Bruhat compatible with M^w by Lemma 5.1 and the entries in purple are Bruhat compatible with M^w by Lemma 5.2.

$$M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & x & & & x & x & x & x & & x \\ \hline & x & x! & & & & & x & x & x & x & & x \\ \hline & x & x! & & & & & x! & & x & & & x \\ \hline & x & x! & & & & & x! & & & & & \\ \hline & x & & & & & & & & & & & \\ \hline \end{array} \xrightarrow{\text{(2,5)-rim hook}} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline x & x & x & x & x & x & x & x & x & x & x & x & x \\ \hline x & x & x & & \downarrow & & & x & x & x & x & & x \\ \hline & x & \uparrow & & x & & & x & x & x & \downarrow & & x \\ \hline & x & \uparrow & & & & & x & & x & x & & \downarrow \\ \hline & x & \uparrow & & & & & x & & & & & x \\ \hline & x & & & & & & & & & & & \\ \hline \end{array}.$$

We use Lemmas 5.1 and 5.2 to produce the inequality stated in the next lemma.

Lemma 5.4. *Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Let $v_0 := v$. Let $\mathcal{C}$ be a chain in W^P in terms of Maya diagrams given by*

$$\mathcal{C} : M^{v_0} \xrightarrow{\mathcal{C}^{(a_0,b_0)}_{(v_0,w)}} M^{v_1} \xrightarrow{\mathcal{C}^{(a_1,b_1)}_{(v_1,w)}} M^{v_2} \xrightarrow{\mathcal{C}^{(a_2,b_2)}_{(v_2,w)}} \cdots \xrightarrow{\mathcal{C}^{(a_{r-1},b_{r-1})}_{(v_{r-1},w)}} M^{v_r}$$

where $M^{v_j} \not\geqslant M^w$ for $1 \leqslant j \leqslant r-1$ and $M^{v_r} \geqslant M^w$. It follows that $Comp_y\mathcal{C} \leqslant \deg_y(v, w)$.

The strategy of the proof of Lemma 5.4 is to show that if the number of (a, b)-rim hooks applied to the yth row of a Maya diagram equals $\deg_y(v, w)$ (as in Definition 4.16), then the corresponding rows are Bruhat compatible. Furthermore, since the rows are Bruhat compatible there is no need to apply another (a, b)-rim hook to that row which gives $\deg_y(v, w)$ as an upper bound to the number of (a, b)-rim hooks needed to apply to the yth row.

Proof. Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Let $v_0 := v$. Let $1 \leqslant D \leqslant r$ be such that $\mathcal{C}*$ is a chain in W^P in terms of Maya diagrams given by

$$\mathcal{C}* : M^{v_0} \xrightarrow{\mathcal{C}^{(a_0,b_0)}_{(v_0,w)}} M^{v_1} \xrightarrow{\mathcal{C}^{(a_1,b_1)}_{(v_1,w)}} M^{v_2} \xrightarrow{\mathcal{C}^{(a_2,b_2)}_{(v_2,w)}} \cdots \xrightarrow{\mathcal{C}^{(a_{D-1},b_{D-1})}_{(v_{D-1},w)}} M^{v_D}$$

where $\mathrm{Comp}_y\mathcal{C}* \leqslant \deg_y(v, w)$. If $m^{v_j}_y \geqslant m^w_y$ for some $1 \leqslant j \leqslant D-1$, then we are done since another (a, b)-rim hook will not be applied to the yth row. Suppose $m^{v_j}_y \not\geqslant m^w_y$ for $1 \leqslant j \leqslant D-1$ and $\mathrm{Comp}_y\mathcal{C}* = \deg_y(v, w)$. Our aim is to show that $m^{v_D}_y \geqslant m^w_y$.

Suppose that the x in $m_y^{v_{D-1}}$ that is not in $m_y^{v_D}$ is in position (y, z_0) and that the new x in $m_y^{v_D}$ that is not in $m_y^{v_{D-1}}$ is in position (y, z_1). By Lemmas 5.1 and 5.2, if $z \leqslant z_0$ or $z_1 \leqslant z$ then $S_y(M^{v_D}, z) \leqslant S_y(M^w, z)$.

Let $z_0 < z' < z_1$. Let $\mathcal{C}(y)$ be a chain in terms Maya diagrams given by

$$\mathcal{C}(y) : M^{\pi_y(v_0)} \xrightarrow{\mathcal{C}^{(1,2)}_{(\pi_y(v_0), \pi_y(w))}} M^{\pi_y(v_1)} \xrightarrow{\mathcal{C}^{(1,2)}_{(\pi_y(v_1), \pi_y(w))}} \cdots \xrightarrow{\mathcal{C}^{(1,2)}_{(\pi_y(v_{\deg_y(v,w)-1}), \pi_y(w))}} M^{\pi_y(v_{\deg_y(v,w)})}.$$

where $M^{\pi_y(v_j)} \ngeqslant M^{\pi_y(w)}$ for $1 \leqslant j \leqslant \deg_j(v, w) - 1$ and $M^{\pi_y(v_{\deg_y(v,w)})} \geqslant M^{\pi_y(w)}$. In the chain $\mathcal{C}*$ the number of x's in row y removed before z_0 is $\deg_y(v, w)$ and they are replaced behind z_1. Similarly, in the chain $\mathcal{C}(y)$ the number of x's in row 1 removed before z_0 is $\deg_y(v, w)$ and they are replaced behind z_1. This implies that $S_y(M^{v_D}, z') = S_1(M^{\pi_y(v_{\deg_y(v,w)})}, z')$. Then

$$S_y(M^{v_D}, z') = S_1(M^{\pi_y(v_{\deg_y(v,w)})}, z') \leqslant S_1(M^{\pi_y(w)}, z') = S_y(M^w, z').$$

Therefore, $m_y^{v_D} \geqslant m_y^w$.

The result follows since $y \notin \{a_i, a_i + 1, \cdots, b_i - 1\}$ for $D \leqslant i \leqslant r - 1$ (that is, we do not apply another (a, b)-rim hook to the yth row). $\square$

6. Main Result

We arrive at our main theorem by observing that $\deg_y(v, w)$ is bounded above and below by $\mathrm{Comp}_y \mathcal{C}$ as stated in Theorem 6.1. We provide an example of calculating the minimum quantum degree using Maya diagrams in Example 6.2 and state a chain that yields a curve of minimum degree.

Theorem 6.1. *Let M^v and M^w be a Maya diagram corresponding to $v, w \in W^P$. Let $v_0 := v$. Let $\mathcal{C}$ be the chain in W^P in terms of Maya diagrams given by*

$$\mathcal{C} : M^{v_0} \xrightarrow{\mathcal{C}^{(a_0,b_0)}_{(v_0,w)}} M^{v_1} \xrightarrow{\mathcal{C}^{(a_1,b_1)}_{(v_1,w)}} M^{v_2} \xrightarrow{\mathcal{C}^{(a_2,b_2)}_{(v_2,w)}} \cdots \xrightarrow{\mathcal{C}^{(a_{r-1},b_{r-1})}_{(v_{r-1},w)}} M^{v_r}$$

where $M^{v_j} \ngeqslant M^w$ for $1 \leqslant j \leqslant r - 1$ and $M^{v_r} \geqslant M^w$. Then

$$(\deg_1(v, w), \cdots, \deg_k(v, w)) = \sum_{j=0}^{r-1} \deg \mathcal{C}^{(a_j,b_j)}_{(v_j,w)}.$$

Proof. By Lemma 5.4, $\mathrm{Comp}_y \mathcal{C} \leqslant \deg_y(v, w)$. By Lemma 4.18 we have that $\deg_y(v, w) \leqslant \mathrm{Comp}_y \mathcal{C}$. Then $\mathrm{Comp}_y \mathcal{C} \leqslant \deg_y(v, w) \leqslant \mathrm{Comp}_y \mathcal{C}$. The result follows. $\square$

Example 6.2. Here we give an example of Theorem 6.1. Consider the following two Maya diagrams.

$$M^w = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & x & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & x & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & x & x \\ \hline {\color{blue}x} & x & x & & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & & \\ \hline {\color{blue}x} & & & & {\color{blue}x} & x & x & & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & & \\ \hline {\color{blue}x} & & & & x & & & & {\color{blue}x} & x & x & & \\ \hline {\color{blue}x} & & & & & & & & x & x & & & \\ \hline x & & & & & & & & & & & & \\ \hline \end{array} \text{ and } M^v = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & x & {\color{blue}x} & x & x & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & x & {\color{blue}x} \\ \hline x & {\color{blue}x} & {\color{blue}x} & & x & & & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & {\color{blue}x} & & {\color{blue}x} \\ \hline & {\color{blue}x} & {\color{blue}x} & & & & & {\color{blue}x} & x & {\color{blue}x} & x & & {\color{blue}x} \\ \hline & {\color{blue}x} & {\color{blue}x} & & & & & {\color{blue}x} & & x & & & x \\ \hline & {\color{blue}x} & x & & & & & x & & & & & \\ \hline & x & & & & & & & & & & & \\ \hline \end{array}.$$

Then we have the following where ! indicate the Bruhat order incompatible positions and $\uparrow$ and $\downarrow$ describe the movement of the x's.

x	x	x	x	x	x	x	x	x	x	x	x	x
x	x	x		x			x	x	x	x		x
	x	$x!$					x	x	x	x		x
	x	$x!$					$x!$		x			x
	x	$x!$					$x!$					
	x											

$\xrightarrow{(2,5)\text{-rim hook}}$

x	x	x	x	x	x	x	x	x	x	x	x	x
x	x	x		$\downarrow$			x	x	x	x		x
	x	$\uparrow$		x			x	x	x	$\downarrow$		x
	x	$\uparrow$					x		x	x		$\downarrow$
	x	$\uparrow$					x					x
	x											

$=$

x	x	x	x	x	x	x	x	x	x	x	x	x
x	x	x		x			x	x	x	x		x
	x			x			x	x	x	x		x
	x						x		x	x		x
	x						$x!$					x
	x											

$\xrightarrow{(2,3)\text{-rim hook}}$

x	x	x	x	x	x	x	x	x	x	x	x	x
x	x	x		x			x	x	x	x		x
	x			x			x	x	x	x		x
	x						x		x	$\downarrow$		x
	x						$\uparrow$			x		x
	x											

$=$

x	x	x	x	x	x	x	x	x	x	x	x	x
x	x	x		x			x	x	x	x		x
	x			x			x	x	x	x		x
	x						x		x	x		x
	x									x		x
	x											

.

Thus the minimum quantum degree that appears in $\sigma^v \star \sigma_w$ is

$$(0,1,1,1,0) + (0,1,0,0,0) = (0,2,1,1,0).$$

Following the process in Example 4.10, the precise chain to describe this curve is given by

$$v s_{e_2-e_5} s_{e_5-e_7} s_{e_8-e_9} s_{e_7-e_8} s_{e_3-e_5}.$$

7. Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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