ON FROBENIUS-PERRON DIMENSION

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Abstract. We propose a notion of Frobenius-Perron dimension for certain free $\mathbb{Z}$-modules of infinite rank and compute it for the $\mathbb{Z}$-modules of finite dimensional complex representations of unitary groups with nonnegative dominant weights. We also provide a lower bound for the Frobenius-Perron dimension of Schubert classes in the quantum cohomology of complex Grassmannians.

1. Introduction

The well-known Frobenius-Perron theory (see e.g. [4]) concerns eigenvalues and eigenvectors of nonnegative square matrices of finite order, and has applications to many areas of research in mathematics. As one remarkable property in Frobenius-Perron theory, the spectral radius of an $N \times N$ irreducible nonnegative matrix is a simple eigenvalue of the matrix. The notion of the Frobenius-Perron dimension was motivated by that of the index of a subfactor [21]. It was first defined for commutative fusion rings by Fröhlich and Kerler [15] as functions satisfying certain properties. The theory of Frobenius-Perron dimensions for general fusion rings and categories was developed by Etingof, Nikshych and Ostrik [12]; for some other cases, we refer to [10, 13, 5, 11, 29] and references therein. The key ingredient here is to define a function on a nice $\mathbb{Z}_+$-ring of finite rank whose value at a basis element is the spectral radius of the induced linear operator as guaranteed by the Frobenius-Perron theory. Such function value is called the Frobenius-Perron dimension of the corresponding basis element. In the present paper, we will build a "Frobenius-Perron bridge", connecting objects of interest in algebraic geometry, representation theory and operator algebras.

One side of the bridge relates with two conjectures in algebraic geometry, which are about eigenvalue problems on quantum cohomology. The quantum cohomology ring $(QH^*(X), \star)$ of a Fano manifold X is a deformation of the classical cohomology ring $(H^*(X), \cup)$. Each element α in $QH^*(X)$ naturally induces a linear operator $\hat{\alpha}$ on it by using the quantum product $\hat{\alpha}(\beta) := \alpha \star \beta$. The conjecture $\mathcal{O}$, proposed by Galkin, Golyshev and Iritani [18], concerns the relationship between the spectral radius $\rho(\hat{c}_1)$ and the eigenvalues of the linear operator $\hat{c}_1$ on a specialization of $QH^*(X)$ at the quantum variables $\mathbf{q}$, where c_1 denotes the first Chern class of X. There is another conjecture proposed by Galkin [17], says that $\rho(\hat{c}_1) \geq \dim_{\mathbb{C}} X + 1$ with equality if and only if X is a complex projective space. Let us move to the important case when X is a flag variety $G_{\mathbb{C}}/P$. Here $G_{\mathbb{C}}$ denotes a connected, complex semisimple Lie group, and P denotes a parabolic subgroup of $G_{\mathbb{C}}$. The specialization $QH^*(G_{\mathbb{C}}/P)|_{\mathbf{q}=\mathbf{1}}$ is a $\mathbb{Z}_+$-ring of finite rank with a standard $\mathbb{Z}_+$-basis of Schubert classes $[X_w]$'s. The conjecture $\mathcal{O}$ for $G_{\mathbb{C}}/P$ was verified by Cheong and the first named author [7] by using Frobenius-Perron theory together with computation of certain Gromov-Witten invariants. Galkin's lower bound conjecture can be rephrased as for the lower bound of the Frobenius-Perron dimension of Schubert divisor classes. This naturally leads to the following question.

Question 1.1. *Can we give an explicit formula or a lower bound on the Frobenius-Perron dimension of a general Schubert class* $[X_w]$ *in* $QH^*(G_{\mathbb{C}}/P)|_{\mathbf{q}=\mathbf{1}}$*?*

The special case of complex Grassmannians can be well studied: Rietsch has given an explicit formula in [25], and we will provide an lower bound in **Theorem 3.7**. We remark that Galkin's lower bound was verified for complex Grassmannians [14] and for Lagrangian and orthogonal Grassmannians [6]. Numerical investigation to Question 1.1 was recently explored for certain flag varieties of Lie type A in [27].

Another side of the bridge turns out to be related with the amenability of dimension problems in representation theory and operator algebras. Analytic properties in the theory of rigid C^*-tensor categories $\mathcal{C}$ have been studied for over thirty years, going back to the work of Ghez-Lima-Roberts [19] and Doplicher-Roberts [8]. A dimension function d on the fusion ring $K(\mathcal{C})$, roughly speaking, is ring homomorphism that takes positive real values at simple objects of $\mathcal{C}$. The notion of the amenability in the context of subfactors was introduced by Popa [24] by equivalent analytic conditions, and was defined for objects in rigid C^*-tensor categories by Long-Roberts [22]. Equivalent conditions of amenability for a pair $(K(\mathcal{C}), d)$, including Folner type characterizations, were given by Hiai-Izumi [20]. As an important example of such categories, we can consider the category $\mathcal{C} = \mathrm{Rep}(G_{\mathrm{cpt}})$ of finite-dimensional complex representations of a compact group G_{cpt}. The positive dimension $d_{\min}$ function on $\mathcal{C}$, obtained by taking standard minimal solutions to the conjugate equations, is amenable. Moreover, the value $d_{\min}([V])$ at an irreducible representation V coincides with the ordinary dimension of the vector space V (see e.g. [23, §2.7] for detailed explanations). Now we may ask the following.

Question 1.2. *Is there an algebraic aspect of* $(\mathrm{Rep}(G_{\mathrm{cpt}}), d_{\min})$*?*

Let us start with the $\mathrm{Rep}(G_{\mathrm{fin}})$ where G_{fin} denotes a finite group. On one hand, such category is finite, on which the intrinsic dimension function is the only dimension function (see e.g. [23, Corollary 2.7.8]), coinciding with the ordinary dimension function on vector spaces. On the other hand, the Frobenius-Perron dimension FPdim of the Grothendieck ring $K(\mathcal{C}) = \mathrm{Gr}(\mathrm{Rep}(G_{\mathrm{fin}}))$ defined as in [12], turns out to be characterized by the following ring homomorphism

$$\mathrm{FPdim} : \mathrm{Gr}(\mathrm{Rep}(G_{\mathrm{fin}})) \to \mathbb{C} \quad \text{with} \quad \mathrm{FPdim}([V_{\mathrm{fin}}]) = \dim V_{\mathrm{fin}}$$

for any irreducible representation V_{fin} of G_{fin}. Thus FPdim serves as the amenable dimension function by the uniqueness. Now there are infinitely many isomorphism classes in the category $\mathrm{Rep}(G)$ for a compact Lie group G (or equivalently, its complexification $G_{\mathbb{C}}$ or its Lie algebra $\mathrm{Lie}(G)$). As a more precise formulation of Question 1.2, we would like to ask the following.

Question 1.3. *Is there an appropriate notion of Frobenius-Perron dimension* FPdim *on* $\mathrm{Gr}(\mathrm{Rep}(G))$*, such that* $d_{\min}$ *can be obtained in terms of specture radius?*

In the present paper, we propose a notion of Frobenius-Perron dimension for $\mathbb{Z}_+^{\bullet}$-rings, generalizing the standard one for $\mathbb{Z}_+$-ring of finite rank. As we will see in Definitions 2.3 and 2.4, we define the generalized Frobenius-Perron dimension to be the limit of the standard one for a $\mathbb{Z}_{\geq 0}$-filtration of $\mathbb{Z}_+$-rings of finite rank. Finite dimensional irreducible representations $\mathbb{S}_\lambda(V)$ of $G = U(k)$ are indexed by decreasing sequences $\lambda = (\lambda_1, \cdots, \lambda_k)$ of integers. The irreducible representations $\mathbb{S}_\lambda(V)$ with $\lambda_k \geq 0$ generate a subcategory $\mathrm{Rep}(U(k))_+$ of $\mathrm{Rep}(U(k))$. We answer Question 1.3 for unitary groups $U(k)$ in the following sense.

Theorem 1.4. *There is a generalized Frobenius-Perron dimension* $\mathrm{FPd}^\bullet$ *of the* $\mathbb{Z}_+^\bullet$*-ring* $\mathrm{Gr}(\mathrm{Rep}(U(k))_+)$*, given by the* $\mathbb{Z}_+^\bullet$*-ring homomorphism*

$$\mathrm{FPd}^\bullet : \mathrm{Gr}(\mathrm{Rep}(U(k))_+) \to \mathbb{C}; \quad \mathrm{FPd}^\bullet([\mathbb{S}_\lambda(V)]) = \dim \mathbb{S}_\lambda(V).$$

We will restate the above conclusion in Theorem 4.3 in a more precise way.

On one hand, the idea of our generalized notion of Frobenius-Perron dimension is natural. The free $\mathbb{Z}$-module $\mathrm{Gr}(\mathrm{Rep}(U(k))_+)$ admits a natural $\mathbb{Z}_{\geq 0}$-filtration of $\mathbb{Z}$-modules $\mathrm{Rep}(U(k))_\ell$ generated by isomorphism classes $[\mathbb{S}_\lambda(V)]$ with $\ell \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Each data $\mathrm{Rep}(U(k))_\ell$ is of finite rank, and inherits a $\mathbb{Z}_+$-ring structure induced from the "natural" tensor product with the help of natural projections. On the other hand, the subtle point here is that we need to consider the "quantum version" of the tensor product instead. Slightly more precisely, we need to consider the fusion ring structure of $\mathrm{Rep}(U(k))_\ell$, called the Verlinde algebra (named after Erik Verlinde [28]) at level $(\ell, k+\ell)$. Those isomorphism classes $[\mathbb{S}_\lambda(V)]$ with $\ell \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ form a $\mathbb{Z}_+$-basis, with the structure constants counting the dimension of the space of sections of appropriate line bundles over the moduli space of semi-stable parabolic bundles, or equivalently, counting the dimension of certain vector spaces of conformal blocks as studied in the physics literature. (See [30, 1, 2, 3] for more details.) We do the computation by using Witten's remarkable ring isomorphism [30, 1, 3]

$$\mathrm{Rep}(U(k))_\ell \to QH^*(Gr(k, k+\ell))|_{q=1}; \quad [\mathbb{S}_\lambda(V)] \mapsto [X_\lambda].$$

Here the Schubert classes $[X_\lambda]$ in the quantum cohomology $QH^*(Gr(k, k+\ell))$ of the complex Grassmannian $Gr(k, k+\ell)$ are indexed by partitions λ. Thanks to the above isomorphism, the computation for the Verlinde algebra can be translated to that for the spectral radius of linear operators on $QH^*(Gr(k, k+\ell))|_{q=1}$ induced by the Schubert classes.

Witten's isomorphism above, together with **Theorem 3.6**, forms the body of our bridge for $G = U(k)$. Although Witten's isomorphism does not exist in general, we can add one more to one side of the bridge by asking the following.

Question 1.5. *What is the Frobenius-Perron dimension of the fusion rings at each level* [2] *of complex simple Lie algebras of general Lie types? If the limit exists, does it give an answer to Question 1.3?*

The paper is organized as follows. In section 2, we review the standard notion of Frobenius-Perron dimension of a $\mathbb{Z}_+$-ring of finite rank and generalize it to that of certain $\mathbb{Z}_+$-ring of infinite rank. In section 3, we study properties of the spectral radius of linear operators on $QH^*(Gr(k, n))|_{q=1}$ induced by Schubert classes. In section 4, we compute the Frobenius-Perron dimension for the polynomial representation ring of unitary groups.

2. Frobenius-Perron dimension

We mainly follow [9] for the standard notion of Frobenius-Perron dimension. A basis $\{\beta_i\}_{i\in I}$ of a ring which is free as a $\mathbb{Z}$-module is called a $\mathbb{Z}_+$-basis if $\beta_i \cdot \beta_j = \sum_{r\in I} c_{ij}^r \beta_r$ with $c_{ij}^r \in \mathbb{Z}_{\geq 0}$ for any i, j, r. Let A be a $\mathbb{Z}_+$-ring, namely a ring with identity 1 together with a fixed $\mathbb{Z}_+$-basis $\{\beta_i\}_{i\in I}$. Each β_i induces a linear operator $\hat{\beta}_i : A \to A; \gamma \mapsto \beta_i \cdot \gamma$. If I is finite, then for $i \in I$, the well-known Frobenius-Perron theory on nonnegative matrices (see e.g. [4, §2, Theorem 1.1]) ensures that the spectral radius $\rho(\hat{\beta}_i)$ of $\hat{\beta}_i$,

$$\rho(\hat{\beta}_i) := \max\{|c| \mid c \text{ is an eigenvalue of } \hat{\beta}_i\} \in \mathbb{R}_{\geq 0},$$

is an eigenvalue of the linear operator $\hat{\beta}_i$.

Definition 2.1. *Let A be a $\mathbb{Z}_+$-ring of finite rank. The function* $\mathrm{FPdim} = \mathrm{FPdim}_A$,

$$\mathrm{FPdim} : A \to \mathbb{C}; \quad \mathrm{FPdim}\left(\sum\nolimits_i a_i\beta_i\right) := \sum\nolimits_i a_i\rho(\hat{\beta}_i),$$

is called the ***Frobenius-Perron dimension*** *of A.*

Furthermore, if A is transitive and unital, then $\mathrm{FPdim} : A \to \mathbb{C}$ is a ring homomorphism, so that the above notion is consistent with that for commutative Fusion rings introduced in [15] . The theory of Frobenius-Perron dimensions for general fusion rings and categories was developed in [12].

Example 2.2. *Let $\mathcal{C} = \mathrm{Rep}(G_{\mathrm{fin}})$ be the category of finite dimensional complex representations of a finite group G_{fin}, and $A = \mathrm{Gr}\big(\mathrm{Rep}(G_{\mathrm{fin}})\big)$ be its Grothendieck ring. Then for any $V \in \mathcal{C}, \mathrm{FPdim}([V]) = \dim_{\mathbb{C}}(V)$.*

In order to explore an answer to Question 1.3, we propose the following definition.

Definition 2.3. *We call a pair $\big(A, \{(A_r, B_r)\}_{r\in\mathbb{Z}_{\geq 0}}\big)$ a $\mathbb{Z}_+^\bullet$-ring (and simply denote it as A), if A is a free $\mathbb{Z}$-module and the family $\{(A_r, B_r)\}_{r\in\mathbb{Z}_{\geq 0}}$ satisfy the following.*

1. *$\{(A_r, +)\}_r$ is a $\mathbb{Z}_{\geq 0}$-filtration of $(A, +)$ of free $\mathbb{Z}$-modules of finite rank;*
2. *$(A_r, +, \star_r)$ is a $\mathbb{Z}_+$-ring with fixed $\mathbb{Z}_+$-basis B_r;*
3. *$B_r \subset B_{r'}$, for any $r, r' \in \mathbb{Z}_{\geq 0}$ with $r < r'$;*
4. *B_0 contains an element 1 which is the identity of $(A_r, \star_r)$ for all r.*

We call a map $F : A \to \mathbb{C}$ a $\mathbb{Z}_+^\bullet$-ring homomorphism, if there exists a family $\{F_r : (A_r, +, \star_r) \to (\mathbb{C}, +, \cdot)\}_r$ of ring homomorphisms in the usual sense such that $\lim\limits_{r\to+\infty} F_r(\alpha) = F(\alpha)$ for any $\alpha \in A$ (where $F_r(\alpha) := 0$ if $\alpha \notin A_r$ for conventions).

Definition 2.4. *Let $\big(A, \{(A_r, B_r)\}_{r\in\mathbb{Z}_{\geq 0}}\big)$ be a $\mathbb{Z}_+^\bullet$-ring. Assume $\lim\limits_{r\to+\infty} \mathrm{FPdim}_{A_r}(\beta)$ exits and belongs to $\mathbb{R}$ for all $\beta \in \bigcup_{r=0}^{\infty} B_r$. We denote $\mathrm{FPd}^\bullet(\beta) := \lim\limits_{r\to+\infty} \mathrm{FPdim}_{A_r}(\beta)$, and call its linear extension $\mathrm{FPd}^\bullet : A \to \mathbb{C}$ the* ***Frobenius-Perron dimension*** *of $\big(A, \{(A_r, B_r)\}_{r\in\mathbb{Z}_{\geq 0}}\big)$ if $\mathrm{FPd}^\bullet$ is a $\mathbb{Z}_+^\bullet$-ring homomorphism.*

Example 2.5. *Let A be a $\mathbb{Z}_+$-ring of finite rank with fixed $\mathbb{Z}_+$-basis B. Then A is a $\mathbb{Z}_+^\bullet$-ring with respect to the trivial $\mathbb{Z}_{\geq 0}$-filtration $\{A_r\}$ defined by $A_r = A$ and $B_r = B$ for any $r \geq 0$. In this case, a $\mathbb{Z}_+^\bullet$-ring homomorphism is a ring homomorphism in the usual sense, and the generalized notion of Frobenius-Perron dimension of A coincides with the standard one as defined in Definition 2.1*

Remark 2.6. *The underlying $\mathbb{Z}$-module A of a $\mathbb{Z}_+^\bullet$-ring is not necessarily a ring a priori.*

Example 2.7. *Let A be a $\mathbb{Z}_+$-ring equipped with $\{(A_r, B_r)\}_r$ such that* (a) *$1_A \in B_0$;* (b) *B_r is an additive basis of A_r; and* (c) *conditions (1) and (3) in Definition*

2.3 are satisfied. Then we can equip A with a $\mathbb{Z}_+^\bullet$-ring structure, by considering the induced $\mathbb{Z}_+$-ring structure on $(A_r, +, \circ_r)$ defined by

$$\alpha \circ_r \beta := \pi_r(\alpha \circ \beta), \quad \forall \alpha, \beta \in A_r.$$

Here $\pi_r : A \to A_r$ is the morphism of $\mathbb{Z}$-modules given by $\pi_r(\beta) = \beta$ if $\beta \in B_r$, or 0 if $\beta \in \bigcup_s B_s \setminus B_r$. In this case, the limit $\lim\limits_{r\to+\infty} \mathrm{FPdim}_{A_r}(\beta)$ exists (possibly equal to $+\infty$) for any $\beta \in B$, by Frobenius-Perron theory (see e.g. [4, §2, Corollary 1.6]*).*

3. Quantum cohomology of Grassmannians

Our generalized notion of Frobenius-Perron dimension is motivated by the study of the quantum cohomology of the complex Grassmannian $Gr(k, n) = \{W \leqslant \mathbb{C}^n \mid \dim W = k\}$, where $k, n \in \mathbb{Z}_{>0}$ with $k < n$. The readers with a strong interest in the computation of generalized Frobenius-Perron dimension for polynomial representation ring of unitary groups, can skip this section.

Set $E_i^{(i)} = \mathbb{C}^i$ and $E_i^{(j)} = E_{i-1}^{(j)} \times \{0\}$ for any $i \le j$. Then $E_\bullet^{(n)} = \{E_1^{(n)} \leqslant E_2^{(n)} \leqslant \cdots \leqslant E_n^{(n)}\}$ is a complete flag in $E_n^{(n)} = \mathbb{C}^n$. Let $\mathcal{P}_k(n) = \{\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{Z}^k \mid n-k \ge \lambda_1 \ge \cdots \ge \lambda_k \ge 0\}$. For $\lambda \in \mathcal{P}_k(n)$, we denote $|\lambda| = \sum_{i=1}^k \lambda_i$ and $\lambda^\vee = (n-k-\lambda_k, \cdots, n-k-\lambda_1)$.

The complex Grassmannian $Gr(k, n)$ has a class of closed subvarieties, called Schubert subvarieties that are defined by

$$X_\lambda = X_\lambda(E_\bullet^{(n)}) = \{W \in Gr(k,n) \mid \dim(W \cap E_{n-k+i-\lambda_i}^{(n)}) \ge i, i = 1, \ldots, k\}$$

where X_λ is of codimension $|\lambda|$ for any partition $\lambda \in \mathcal{P}_k(n)$. The cohomology classes $\{[X_\lambda] \in H^{2|\lambda|}(Gr(k,n), \mathbb{Z})\}$ form a $\mathbb{Z}_+$-basis of the integral cohomology ring $H^*(Gr(k,n)) = H^*(Gr(k,n), \mathbb{Z})$, which is torsion free and of rank $\binom{n}{k}$.

The (small) quantum cohomology $QH^*(Gr(k,n)) = (H^*(Gr(k,n)) \otimes \mathbb{Z}[q], \star)$ is a commutative ring with the quantum product for any $\lambda, \mu \in \mathcal{P}_k(n)$ defined by

$$[X_\lambda] \star [X_\mu] = \sum_{\nu \in \mathcal{P}_k(n), d \in \mathbb{Z}_{\ge 0}} N_{u,v}^{w,d} [X_\nu] q^d.$$

Here the Schubert structure constant $N_{u,v}^{w,d}$, known as a genus 0, 3-point Gromov-Witten invariant, counts the number of holomorphic maps $f : \mathbb{P}^1 \to Gr(k,n)$ of degree d with $f(0) \in X_\lambda, f(1) \in g \cdot X_\mu$ and $f(\infty) \in g' \cdot X_{\nu^\vee}$ for generic (fixed) $g, g' \in GL(n, \mathbb{C})$. In particular, $N_{\lambda,\mu}^{\nu,d}$ is a non-negative integer for any d, and it vanishes for any sufficient large d.

For any partition $\lambda \in \mathcal{P}_k(n)$, the quantum product by the Schubert class $[X_\lambda]$ induces a linear operator

$$\widehat{[X_\lambda]} : QH^*(Gr(k,n))|_{q=1} \longrightarrow QH^*(Gr(k,n))|_{q=1};\ \beta \mapsto [X_\lambda] \star \beta|_{q=1}.$$

Its eigenvalues and eigenvectors have been well studied by Rietsch [25] in terms of Schur functions and primitive n-th roots of unity. In particular, the following lemma follows immediately from Theorem 8.4 (1) and section 11 of [25]

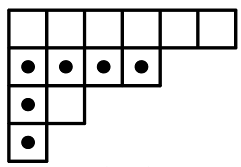
Lemma 3.1. *Let $\lambda \in \mathcal{P}_k(n)$, and let $\rho_{k,\lambda}(n)$ denote the spectral radius of $\widehat{[X_\lambda]}$, namely*

$$\rho_{k,\lambda}(n) = \max\{|c| \mid c \textit{ is an eigenvalue of the operator } \widehat{[X_\lambda]} \textit{ on } QH^*(Gr(k,n))|_{q=1}\}.$$

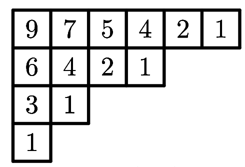
Denote $\mathrm{hl}(i,j)=\lambda_i+\lambda_j^t-i-j+1$, *where* $\lambda^t=(\lambda_1^t,\cdots,\lambda_{n-k}^t)$ *denotes the transpose partition of* λ. *Then we have*

$$\rho_{k,\lambda}(n)=\frac{\prod_{(i,j)\in\lambda}\sin\left((k-i+j)\frac{\pi}{n}\right)}{\prod_{(i,j)\in\lambda}\sin\left(\mathrm{hl}(i,j)\frac{\pi}{n}\right)}, \tag{3.1}$$

Example 3.2. *The number* $\mathrm{hl}(i,j)$ *equals the hook length of the box labeled by* (i,j) *in the Young diagram of the partition* λ. *The following figure shows the case* $\lambda=(6,4,2,1)\in\mathcal{P}_4(11)$, *for which* $\lambda^t=(4,3,2,2,1,1,0)$.



$\mathrm{hl}(2,1)=6$



$\mathrm{hl}(i,j)$

Remark 3.3. *The number* $\rho_{k,(1,0,\cdots,0)}(n)$ *is the length of the* k*-th diagonal of a regular* n *sided polygon with unit side length.*

Notice that there are $|\lambda|$ boxes in the i-th row of the Young diagram of the partition λ. The sequence $\{k-i+j\}_{(i,j)\in\lambda}$ of length $|\lambda|$ can be reordered as $\{a_1,\cdots,a_{|\lambda|}\}$ such that $a_r=k-i+\lambda_i-j+1$ correspondes to a unique $(i,j)\in\lambda$. For the same (i,j), we denote $b_r=\mathrm{hl}(i,j)$. Then $a_r\geq b_r$ for all $1\leq r\leq|\lambda|$, since

$$k-i+\lambda_i-j+1\geq\lambda_j^t+\lambda_i-i-j+1=\mathrm{hl}(i,j),\ \text{for any }(i,j)\in\lambda.$$

By using formula (3.1), we can define a smooth function $\rho_{k,\lambda}:\mathbb{R}^+\to\mathbb{R}$. The next property is an immediate consequence of the definitions of a_r's and b_r's.

Corollary 3.4. *For any* $\lambda\in\mathcal{P}_k(n)$, *the following are equivalent.*

i) $\rho_{k,\lambda}(x)=1$; ii) $a_r=b_r$ for all $1\leq r\leq|\lambda|$; iii) $\lambda_1=\lambda_2=\cdots=\lambda_k$.

Lemma 3.5 (Hook-length formula; see page 61 of [26])**.** *Let* $\mathbb{S}_\lambda(V)$ *denote the irreducible representation of* $U(k)$ *associated to* λ. *Then*

$$\dim\mathbb{S}_\lambda(V)=\frac{\prod_{(i,j)\in\lambda}(k-i+j)}{\prod_{(i,j)\in\lambda}\mathrm{hl}(i,j)}.$$

Theorem 3.6. *For* $\lambda\in\mathcal{P}_k(n)$, *we have*

$$\lim_{x\to+\infty}\rho_{k,\lambda}(x)=\dim\mathbb{S}_\lambda(V).$$

Proof. The statement follows immediately from and Lemmas 3.1 and 3.5. □

Theorem 3.7. *For* $\lambda\in\mathcal{P}_k(n)$ *with* $\lambda_i\neq\lambda_j$ *for some* $i\neq j$, *the following hold.*

(1) *We have the inequality*

$$\rho_{k,\lambda}(n)\geq\dim\mathbb{S}_\lambda(V)\prod_{(i,j)\in\lambda}\left(1-\frac{\pi^2(k-i+j)^2}{6n^2}\right).$$

(2) *The function* $\rho_{k,\lambda}(x)$ *is strictly increasing on the interval* $(k+\lambda_1-1,+\infty)$, *and is concave down when* x *is sufficiently large.*

Proof. We leave the details in the appendix. □

Remark 3.8. *The above lower bound of $\rho_{k,\lambda}(n)$ works for any $1 \le k < n$ and $\lambda \in \mathcal{P}_k(n)$. There is a natural isomorphism $QH^*(Gr(k,n)) \longrightarrow QH^*(Gr(n-k,n))$ that sends every Schubert class $[X_\lambda]$ for $Gr(k,n)$ to the Schubert class $[X_{\lambda^t}]$ of $Gr(k,n)$ labeled by the transpose λ^t of the partition $\lambda \in \mathcal{P}_k(n)$. As a consequence, we have*

$$\rho_{k,\lambda}(n) = \rho_{n-k,\lambda^t}(n) \ge \dim \mathbb{S}_\lambda(V^t) \prod_{(i,j)\in\lambda^t} \left(1 - \frac{\pi^2(n-k-i+j)^2}{6n^2}\right)$$

where $\mathbb{S}_{\lambda^t}(V^t)$ denotes the irreducible representation of $U(n-k)$ associated to λ^t.

Example 3.9. *For $k = 2$, we have $\dim \mathbb{S}_\lambda(V) = \lambda_1 - \lambda_2 + 1$. Moreover, by simplifying formula* (3.1)*, we obtain $\rho_{2,\lambda}(n) = \dfrac{\sin \frac{(\lambda_1-\lambda_2+1)\pi}{n}}{\sin \frac{\pi}{n}}$. It follows from Theorem 3.7 (1) that $\rho_{2,\lambda}(n) \ge (\lambda_1 - \lambda_2 + 1)\left(1 - \frac{\pi^2(\lambda_1+1)^2}{6n^2}\right)$. In particular,*

$$\rho_{2,(1,0)}(n) \ge 2\left(1 - \frac{4\pi^2}{6n^2}\right) > 2\left(1 - \frac{3}{2n}\right) = k\frac{n-k}{n} + \frac{1}{n} \quad \text{for } n > 4,$$

$\rho_{2,(1,0)}(3) = \frac{\sin\frac{2\pi}{3}}{\sin\frac{\pi}{3}} = 1 = 2\frac{3-2}{3} + \frac{1}{3}$, *and* $\rho_{2,(1,0)}(4) = \frac{\sin\frac{2\pi}{4}}{\sin\frac{\pi}{4}} = \sqrt{2} > 2\frac{4-2}{4} + \frac{1}{4}$.

Example 3.10. *For $\lambda = (1, 0, \cdots, 0)$, we notice $\dim \mathbb{S}_\lambda(V) = k$ so that*

$$\rho_{k,\lambda}(n) \ge k\left(1 - \frac{\pi^2 k^2}{6n^2}\right) > k\frac{n-k}{n} + \frac{1}{n} \text{ for } 3 \le k \le \frac{n}{2}$$

by Theorem 3.7 (1). Together with Remark 3.8 and Example 3.9, this show that

$$n\rho_{k,(1,0,\cdots,0)}(n) \ge \dim Gr(k,n) + 1 = k(n-k) + 1$$

with the equality holding if and only if $k = 1$ or $n - 1$. This is exactly the proof of Galkin's lower bound conjecture for all $Gr(k,n)$ as was given in [14].

4. Frobenius-Perron dimension of polynomial representation ring of unitary groups

As in the introduction, we consider the Grothendieck ring $\mathrm{Gr}\big(\mathrm{Rep}(U(k))_+\big)$ of the subcategory $\mathrm{Rep}(U(k))_+$ of finite dimensional complex representations of $U(k)$ generated by the isomorphism classes of irreducible representations $\mathbb{S}_\lambda(V)$ with $\lambda \in \bigcup_{r=0}^\infty \mathcal{P}_k(k+r)$, namely $\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{Z}^k$ satisfying $\lambda_1 \ge \cdots \lambda_k \ge 0$. The Grothendieck ring $A = \mathrm{Gr}\big(\mathrm{Rep}(U(k))_+\big)$, also referred to as the polynomial representation ring of $U(k)$, is a $\mathbb{Z}_+$-ring of infinite rank. Moveover, it admits a standard $\mathbb{Z}_{\ge 0}$-filtration $\{A_r\}_r$ of free $\mathbb{Z}$-modules, with $B_r := \{[\mathbb{S}_\lambda(V)] \mid \lambda \in \mathcal{P}_k(k+r)\}$ being a basis of A_r for each r. As to be described below, the $\mathbb{Z}_{\ge 0}$-filtration $\{A_r\}_r$ can be equipped with different ring structures, for which A becomes a $\mathbb{Z}_+^\bullet$-ring.

4.1. Ring structures induced from the tensor product.

Consider the morphism π_r of $\mathbb{Z}$-modules given by $\pi_r : A \to A_r$ with $\pi_r([\mathbb{S}_\lambda(V)]) = [\mathbb{S}_\lambda(V)]$ if $\lambda \in \mathcal{P}_k(k+r)$, or 0 otherwise. Notice that the ring structure of A is given by tensor product of representations. This induces a natural ring structure $(A_r, \circ_r)$ by

$$[\mathbb{S}_\lambda(V)] \circ [\mathbb{S}_\mu(V)] = \pi_r([\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V)]), \quad \forall \lambda, \mu \in \mathcal{P}_k(k+r).$$

Then $(A_r, \circ_r)$ is a $\mathbb{Z}_+$-ring of finite rank with the identity $\pi_r([\mathbf{1}])$, where $\mathbf{1}$ denote the trivial representation of $U(k)$, whose isomorphic class is the identity of A. Consequently, the limit $\mathrm{FPd}^\bullet([\mathbb{S}_\lambda(V)]) = \lim_{r\to+\infty} \mathrm{FPdim}_{A_r}([\mathbb{S}_\lambda(V)])$ exists for any λ with $\lambda_k \geq 0$ (for instance by [4, §2, Corollary 1.6]). More is true: there is a ring isomorphism (see e.g. [16, section 9.4])

$$(A_r, \circ_r) \longrightarrow (H^*(Gr(k, k+r)), \cup); [\mathbb{S}_\lambda(V)] \mapsto [X_\lambda].$$

For any $\lambda \neq \mathbf{0}$, the cup product of any cohomology class by $[X_\lambda]$ increases the degree, and hence the operator $[X_\lambda]\cup$ is nilpotent. Therefore for any r and any $\lambda \in \mathcal{P}_k(k+r)$, we have $\mathrm{FPd}^\bullet([\mathbb{S}_\lambda(V)]) = \mathrm{FPdim}_{A_r}([\mathbb{S}_\lambda(V)]) = 1$ if $\lambda = \mathbf{0}$, or 0 otherwise. In a summary, we have the following.

Theorem 4.1. *The $\mathbb{Z}$-module A= $\mathrm{Gr}\big(\mathrm{Rep}(U(k))_+\big)$ equipped with the family $\{(A_r, B_r)\}_r$ is a $\mathbb{Z}_+^\bullet$-ring. The family $\{\mathrm{FPdim}_{A_r} : A_r \to \mathbb{C}\}_r$ and the resulting in map $\mathrm{FPd}^\bullet$: $(A, \circ) \to \mathbb{C}$ are all trivial ring homomorphisms. In particular, $\mathrm{FPd}^\bullet$ is the Frobenius-Perron dimension of the $\mathbb{Z}_+^\bullet$-ring $\big(A, \{(A_r, B_r)\}_r\big)$.*

4.2. **Verlinde algebras.** The free $\mathbb{Z}$-module A_r can be equipped with the fusion ring structure $\star_r$ at level $(r, k+r)$, called the Verlinde algebra (of $U(k)$) at level $(r, k+r)$ in the physics literature. We follow [3] for the description of $\star_r$ below.

The irreducible representation $\mathbb{S}_\lambda(V)$ of $U(k)$ restrict to the irreducible representation $V_{\bar\lambda}$ of $SU(k)$, where $\bar\lambda = (\lambda_1 - \lambda_2, \cdots, \lambda_{k-1} - \lambda_k)$. The dual representation $V_{\bar\lambda}^*$ is irreducible and hence given by $V_{\bar\lambda^*}$ for a corresponding dominant weight $\bar\lambda^*$. The fusion ring of $SU(k)$ at level r, denoted by $R(SU(k))_r$, is an associated, commutative ring, defined by

$$[V_{\bar\lambda}] *_r [V_{\bar\mu}] = \sum_{\bar\nu} N_0^{(r)}(\bar\lambda, \bar\mu, \bar\nu)[V_{\bar\nu^*}]$$

where let $N_0^{(r)}(\bar\lambda, \bar\mu, \bar\nu)$ denotes the dimension of the corresponding vector space of conformal blocks for genus 0 at level r as described in [2]. The tensor product of $R(SU(k))_r$ and the fusion ring $R(U(1))_{(k+r)k} = \mathbb{Z}[x]/(x^{k(k+r)} - 1)$ contains a unital subring $\tilde R \leqslant R(SU(k))_r \otimes_{\mathbb{Z}} R(U(1))_{(k+r)k}$ spanned by elements of the form $[V_{\bar\lambda}] \otimes x^a$ where $|\bar\lambda| \equiv a \,(\mathrm{mod}\ k)$. Denote by $\eta_\lambda := (r + \lambda_k, \lambda_1, \cdots, \lambda_{k-1})$. The $\mathbb{Z}$-submodule $\mathcal{I}$ spanned by $\{[V_{\bar\eta_\lambda}] \otimes x^{a+k+r} - [V_{\bar\lambda}] \otimes x^a \mid |\bar\lambda| \equiv a \,(\mathrm{mod}\ k)\}$ is in fact an ideal of $\tilde R$. Then the fusion ring $(A_r, \star_r)$ (i.e. the fusion ring of $U(k)$ at level $(r, r+k)$) can be revealed as the quotient ring $\tilde R/\mathcal{I}$, by identifying $[\mathbb{S}_\lambda(V)] \in A_r$ with $[V_{\bar\lambda}] \otimes x^{|\lambda|} + \mathcal{I} \in \tilde R/\mathcal{I}$. The following remarkable property is due to Witten [30] (see [1, 3] for mathematical proofs).

Proposition 4.2. *The natural isomorphism of $\mathbb{Z}$-modules*

$$\Phi_r : (QH^*(Gr(k, k+r))|_{q=1}, \star) \longrightarrow (A_r, \star_r); \quad [X_\lambda] \mapsto [\mathbb{S}_\lambda(V)]$$

is an isomorphism of rings.

Combining the above descriptions with Theorem 3.6, we have the following.

Theorem 4.3. *The $\mathbb{Z}$-module $A = \mathrm{Gr}\big(\mathrm{Rep}(U(k))_+\big)$ equipped with the family $\{((A_r, \star_r), B_r)\}_r$ is a $\mathbb{Z}_+^\bullet$-ring. The Frobenius-Perron dimension $\mathrm{FPd}^\bullet : \mathrm{A} \to \mathbb{C}$ is well defined (which is a $\mathbb{Z}_+^\bullet$-ring homomorphism), and is given by*

$$\mathrm{FPd}^\bullet : A \to \mathbb{C}; \quad \mathrm{FPd}^\bullet([\mathbb{S}_\lambda(V)]) = \dim \mathbb{S}_\lambda(V).$$

Proof. Notice that $B_r = \{[\mathbb{S}_\lambda(V)] \mid \lambda \in \mathcal{P}_k(r)\} \subset B_{r+1}$ is a $\mathbb{Z}_+$-basis of A_r, and that $[\mathbf{1}] = [\mathbb{S}_{\mathbf{0}}(V)] \in A$ is the common identity of all $(A_r, \star_r)$. Therefore the first statement holds.

Let $n = k + r$. By Proposition 4.2 and Theorem 3.6, we have

$$\mathrm{FPd}^\bullet([\mathbb{S}_\lambda(V)]) = \lim_{r\to+\infty} \mathrm{FPdim}_{A_r}([\mathbb{S}_\lambda(V)]) = \lim_{n\to+\infty} \rho_{k,\lambda}(n) = \dim \mathbb{S}_\lambda(V).$$

Notice that the linear operators $\{\widehat{[X_\lambda]}\}_\lambda$ on $QH^*(Gr(k,n))|_{q=1}$ are commutative. In fact, they can be simultaneously diagonalized [25, section 11] (with respect to the common basis $\{\sigma_I\}$ therein). Hence, the following equalities of spectral radius $\rho_n(\cdot)$ for linear operators on $QH^*(Gr(k,n))|_{q=1}$ hold.

$$\rho_n(a\widehat{[X_\lambda]} + b\widehat{[X_\mu]}) = a\rho_n(\widehat{[X_\lambda]}) + b\rho_n(\widehat{[X_\mu]}); \quad \rho_n(\widehat{[X_\lambda]}\widehat{[X_\mu]}) = \rho_n(\widehat{[X_\lambda]})\rho_n(\widehat{[X_\mu]}).$$

Together with Proposition 4.2, this shows that $\mathrm{FPdim}_{A_r} : A_r \to \mathbb{C}$ is a ring homomorphism for any r. Hence, $\mathrm{FPd}^\bullet : A \to \mathbb{C}$ is a $\mathbb{Z}_+^\bullet$-ring homomorphism. (Here we notice that the way of defining $\mathrm{FPd}^\bullet : A \to \mathbb{C}$ by the linear extension of the map $\mathrm{FPd}^\bullet : B \to \mathbb{C}$ is consistent with the way obtained by taking the limit $\lim\limits_{n\to+\infty} \rho_n(\Phi_r^{-1}(\alpha))$ for any $\alpha \in A$.) □

5. Appendix: Proof of Theorem 3.7

Lemma 5.1. *Let $a > b > 0$, and define $f : \mathbb{R}^+ \to \mathbb{R}; f(x) := \frac{\sin\frac{a}{x}}{\sin\frac{b}{x}}$. Then we have $f'(x) > 0$ for any $x > \frac{a}{\pi}$.*

Proof. By direct calculations, we have

$$f'(x) = \frac{g(x)}{x^2\left(\sin\frac{b}{x}\right)^2}, \quad \text{where} \quad g(x) = -a\cos\frac{a}{x}\sin\frac{b}{x} + b\sin\frac{a}{x}\cos\frac{b}{x}.$$

Therefore $g'(x) = \dfrac{\sin\frac{a}{x}\sin\frac{b}{x}}{x^2}\cdot(-a^2+b^2) < 0$ for any $x > \frac{a}{\pi}$. Notice $\lim\limits_{x\to+\infty} g(x) = 0$. It follows that $g(x) > 0$ and hence $f'(x) > 0$ for any $x \in \left(\frac{a}{\pi}, +\infty\right)$. □

Proof of Theorem 3.7. Notice that $\rho_{k,\lambda}(x) = \prod_{r=1}^{|\lambda|} C_r(x)$ with $C_r(x) = \frac{\sin\frac{a_r\pi}{x}}{\sin\frac{b_r\pi}{x}}$. Clearly $a_r \geq b_r > 0$ for any r, and $\max\{a_r \mid 1 \leq r \leq |\lambda|\} = k + \lambda_1 - 1$.

To prove statement (1), following the proof of [14, Lemma 5.1], we use the elementary inequalities $x - \frac{x^3}{6} \leq \sin x \leq x$ for $x \geq 0$. It follows that

$$C_r(n) = \frac{\sin\frac{a_r\pi}{n}}{\sin\frac{b_r\pi}{n}} \geq \frac{\frac{a_r\pi}{n}\left(1 - \frac{1}{6}\cdot\left(\frac{a_r\pi}{n}\right)^2\right)}{\frac{b_r\pi}{n}} = \frac{a_r}{b_r}\left(1 - \frac{\pi^2 a_r^2}{6n^2}\right),$$

where $a_r \leq k + \lambda_1 - 1 \leq n - 1$ so that $1 - \frac{\pi^2 a_r^2}{6n^2} > 0$. Thus we have

$$\rho_{k,\lambda}(n) = \prod_r C_r(n) \geq \prod_r \frac{a_r}{b_r}\prod_r\left(1 - \frac{\pi^2 a_r^2}{6n^2}\right) = \dim \mathbb{S}_\lambda(V)\prod_{(i,j)\in\lambda}\left(1 - \frac{\pi^2(k-i+j)^2}{6n^2}\right).$$

To prove statement (2), we notice that for any $x > k + \lambda_1 - 1$, $C_r(x) > 0$; moreover, we have $C_r'(x) > 0$ whenever $a_r > b_r$ by Lemma 5.1. Since $\lambda_i \neq \lambda_j$ for some $i \neq j$, $a_r > b_r$ does hold for some r. It follows that

$$\rho_{k,\lambda}'(x) = \rho_{k,\lambda}(x)\sum_{r=1}^{|\lambda|}\frac{C_r'(x)}{C_r(x)} > 0 \text{ for any } x > k + \lambda_1 - 1,$$

where $C_s(x)=1$ and $C_s'(x)=0$ whenever $a_s=b_s$. That is, the first half holds.

$$\begin{aligned}\rho''_{k,\lambda}(x) &= \rho_{k,\lambda}(x)\sum_{i,j}\frac{C_i'(x)C_j'(x)}{C_i(x)C_j(x)} + \rho_{k,n}(x)\sum_r \frac{C_r''(x)}{C_r(x)} - \rho_{k,n}(x)\sum_r\left(\frac{C_r'(x)}{C_r(x)}\right)^2\\ &= \rho_{k,n}(x)\left(\sum_{i\neq j}\frac{C_i'(x)C_j'(x)}{C_i(x)C_j(x)} + \sum_r\frac{C_r''(x)}{C_r(x)}\right).\end{aligned}$$

By Taylor expansion around $x=+\infty$, we have

$$\frac{C_r'(x)}{C_r(x)} = \frac{(-a_r^2+b_r^2)\pi^2}{3x^3} + o\left(\frac{1}{x^4}\right), \quad \frac{C_r''(x)}{C_r(x)} = \frac{(-a_r^2+b_r^2)\pi^2}{x^4} + o\left(\frac{1}{x^5}\right).$$

Hence,

$$x^4\cdot\frac{\rho''_{k,\lambda}(x)}{\rho_{k,n}(x)} = \sum_r(-a_r^2+b_r^2)\pi^2 + o\left(\frac{1}{x}\right),$$

where the leading term is negative. Thus $\rho''_{k,n}(x)<0$ for sufficiently large x. $\square$

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