MINIMUM QUANTUM DEGREES FOR ISOTROPIC GRASSMANNIANS IN TYPES B AND C

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Abstract. We give a formula in terms of Young diagrams to calculate the minimum positive integer d such that q^d appears in the quantum product of two Schubert classes for the submaximal isotropic Grassmannians in Types B and C. We do this by studying curve neighborhoods. We compute curve neighborhoods in several combinatorial models including k-strict partitions and a set of partitions where their inclusion is compatible with the Bruhat order.

1. Introduction

Let $\mathrm{Gr} := \mathrm{Gr}(k,n)$ be the Grassmannian, $\mathrm{IG} := \mathrm{IG}(k,2n)$ be the symplectic Grassmannian, and $\mathrm{OG} := \mathrm{OG}(k,2n+1)$ be the odd orthogonal Grassmannian. Let $\mathrm{QH}^*(X)$ be the quantum cohomology ring of $X \in \{\mathrm{Gr}, \mathrm{IG}, \mathrm{OG}\}$. The purpose of this paper is to give a formula in terms of inclusions of Young diagrams to calculate the minimum positive integer d such that q^d appears in the quantum product of two Schubert classes in $\mathrm{QH}^*(X)$.

The minimum quantum degree was calculated using Young diagrams for Gr by Postnikov [13, 14] and Fulton and Woodward [12]. See also [2, 8, 15]. In a recent paper Bärligea [1] gives an explicit formula to compute the minimum degree of the point class times the point class in terms of the cascade of orthogonal roots in G/P. We approach this problem by considering curve neighborhoods and various combinatorial models.

We study two sets of partitions that index Schubert varieties of OG and IG. The first is the set of $(n-k)$-strict partitions given by

$$\Lambda := \{(\lambda_1 \geq \ldots \geq \lambda_k) : 2n-k \geq \lambda_1, \lambda_k \geq 0, \text{ and if } \lambda_j > n-k \text{ then } \lambda_{j+1} < \lambda_j\}.$$

The $(n-k)$-strict partitions are defined and used to calculate a Pieri rule for isotropic Grassmannians in [9, 10]. For the next set of partitions first recall the set of partitions that index Schubert varieties (by codimension) in the Grassmannian Gr. It is the set $\mathcal{P}(k,n)$ given by

$$\{(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) : n-k \geq \lambda_1, \lambda_k \geq 0\}.$$

The Schubert varieties of OG and IG are indexed by a subset of $\mathcal{P}(k,2n)$. We give the relevant definitions next.

Definition 1.1. Let $\lambda \in \mathcal{P}(k,n)$ be a partition. This partition's boundary consists of n steps moving either left or down in the south-west direction. Let $D(\lambda)(i) = 0$ if the ith step is left and $D(\lambda)(i) = 1$ if the ith step is down.

Definition 1.2. Define $\mathcal{P}'(k,2n) = \{\lambda \in \mathcal{P}(k,2n) :$ if $D(\lambda)(i) = D(\lambda)(2n+1-i)$ then $D(\lambda)(i) = 0\}$

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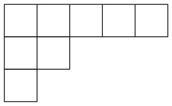
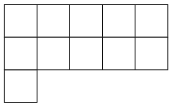
Example 1.3. The partition $(5,2,1) \in \mathcal{P}'(3,8)$. Pictorially,



$\in \mathcal{P}'(3,8)$.

On the contrary, the partition $(5,5,1) \notin \mathcal{P}'(3,8)$. Pictorially,



$\notin \mathcal{P}'(3,8)$.

The partitions Λ and $\mathcal{P}'(k,2n)$ each have a desired property. When using $(n-k)$-strict partitions Λ, the codimension of the Schubert variety X^λ in IG and OG is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. While partition inclusion and the Bruhat order are compatible for partitions in Λ in the cominscule cases (i.e. $k = n$), the compatibility fails to hold in the submaximal cases. On the other hand, partition inclusion on the set $\mathcal{P}'(k,2n)$ *does* respect the Bruhat order (this is Proposition 6.2) but the partitions do not index the (co)dimension of Schubert varieties.

Next we discuss curve neighborhoods. Let $X \in \{\mathrm{Gr}, \mathrm{IG}, \mathrm{OG}\}$. Let $X^\lambda \subset X$ be a Schubert variety where λ is chosen from the appropriate indexing set. The curve neighborhood of X^λ, denoted by $\Gamma_d^X(X^\lambda)$, is the closure of rational curves of degree d that intersect X^λ. It is known from [5] that $\Gamma_d^X(X^\lambda)$ is a Schubert variety. So, there exists a partition λ^d such that $\Gamma_d^X(X^\lambda) = X^{\lambda^d}$. Curve neighborhoods are calculated in terms of Young diagrams in [4, Subsection 3.2, Table 1] for cominuscule Grassmannians. Curve neighborhoods for the submaximal cases are calculated in terms of the Young diagrams in Theorem 5.18 for IG and Theorem 5.20 for OG. These calculations are completed by considering a recursively defined formula for curve neighborhoods that is in terms of roots, coroots, and Weyl group elements. This recursion is given by Buch and Mihalcea in [7] and it is stated in Proposition 2.3.

Example 1.4. For an example of Theorem 5.18 let $\lambda \in P'(k,2n)$ index the Schubert variety $X^\lambda \subset \mathrm{IG}$ and let d be an effective degree. Then $\Gamma_d^{\mathrm{IG}}(X^\lambda) = X^{\lambda^d}$ where

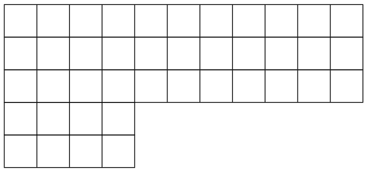
1. the curve neighborhood is indexed by $\lambda^1 = (\lambda_2 - 1 \geq \lambda_3 - 1 \geq \cdots \geq \lambda_k - 1 \geq 0)$ for $d = 1$;
2. and the curve neighborhood is given by the (recursively defined) index $\lambda^d = (\lambda^{d-1})^1$ for $d > 1$.

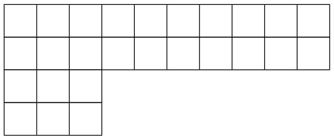
We are using the indexing sets Λ and $\mathcal{P}'(k,2n)$ for the first time to calculate curve neighborhoods and minimum degrees in $\mathrm{IG}(k,2n)$ and $\mathrm{OG}(k,2n+1)$. Thus our main results are a specialization of the results from Fulton and Woodward in [12]. We are ready to state our main results.

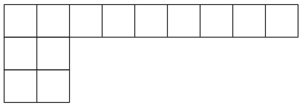
Theorem 1.5. *Let $X \in \{\mathrm{IG}(n,2n), \mathrm{OG}(n,2n+1)\}$. For any pair of Schubert classes $[X^\lambda], [X_\mu] \in \mathrm{QH}^*(X)$ where $\lambda, \mu \in \Lambda$, the smallest degree d such that q^d appears in $[X^\lambda] \star [X_\mu]$ with nonzero coefficient is the smallest integer d such that $\lambda^d \subset \mu$.*

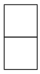
Theorem 1.6. *Let $X \in \{\mathrm{OG}(k,2n+1), \mathrm{IG}(k,2n)\}$. Here $k < n$ for OG. For any pair of Schubert classes $[X^\lambda], [X_\mu] \in \mathrm{QH}^*(X)$ where $\lambda, \mu \in \mathcal{P}'(k,2n)$, the smallest degree d such that q^d appears in $[X^\lambda] \star [X_\mu]$ with nonzero coefficient is the smallest integer d such that $\lambda^d \subset \mu$.*

Example 1.7. For $\mathrm{IG}(5, 2\cdot 8)$ consider $\lambda = (11 \geq 11 \geq 11 \geq 4 \geq 4) \in \mathcal{P}'(5, 2\cdot 8)$ and $\mu = (7,7) \in \mathcal{P}'(5, 2\cdot 8)$. Then we have that

$\lambda =$  and $\mu =$ 

$\lambda^1 =$ 

$\lambda^2 =$ 

$\lambda^3 =$ .

Thus, 3 is the smallest degree d such that q^d appears in $[X^\lambda] \star [X_\mu]$ with nonzero coefficient.

We will discuss the Type A Grassmannian case in its entirety since the proofs of the submaximal cases in Types B and C will follow a similar structure. The $\mathrm{Gr}(k,n), \mathrm{IG}(n, 2n)$, and $\mathrm{OG}(n, 2n+1)$ cases are given in Section 3. In Section 6 we prove Theorem 1.6 for the submaximal isotropic Grassmannian cases in Types B and C.

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2. BACKGROUND

2.1. **Flag varieties.** We will follow the exposition of [6]. Let $X = G/P$ be the flag variety defined by a connected semisimple complex group G and a parabolic subgroup P. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B \cap B^- = T$. Let $W = N_G(T)/T$ be the Weyl group of G, $W_P = N_P(T)/T$ the Weyl group of P, and let $W^P \subset W$ be the subset of minimal-length representatives of the cosets in W/W^P. Let Φ denote the root system of G, with positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$. The parabolic subgroup P is determined by the subset $\Delta_P = \{\beta \in \Delta : s_\beta \in W_P\}$. Each element $w \in W$ defines a B-stable Schubert variety $X_w = \overline{Bw.P}$ and a B^--stable (opposite) Schubert variety $X^w = \overline{B^-w.P}$. If $w \in W^P$ is a minimal-length representative, then $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$.

The group $H_2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module, with a basis consisting of the Schubert classes $[X_{s_\beta}]$ for $\beta \in \Delta \backslash \Delta_P$. Given two elements $d = \sum_\beta d_\beta [X_{s_\beta}]$ and $d' = \sum_\beta d'_\beta [X_{s_\beta}]$ expressed in this basis, we write $d \leq d'$ if and only if $d_\beta \leq d'_\beta$ for each $\beta \in \Delta \backslash \Delta_P$. This defines a partial order on $H_2(X, \mathbb{Z})$.

For any root $\alpha \in \Phi$ that is not in the span Φ_P of Δ_P, there exists a unique irreducible T-invariant curve $X(\alpha) \subset X$ that connects the points $1.P$ and $s_\alpha.P$. An arbitrary irreducible T-invariant curve $C \subset X$ has the form $C = w.X(\alpha)$ for some $w \in W$ and $\alpha \in \Phi^+ \backslash \Phi_P$.

2.2. **Quantum cohomology.** Given an effective degree $d \geq 0$ in $H_2(X,\mathbb{Z})$, we let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps $f : C \to X$ of arithmetic genus zero and degree $f_*[C] = d$. This space is equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$ for $1 \leq i \leq n$, where ev_i sends a stable map to the image of the i-th marked point in its domain. Given cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(X,\mathbb{Z})$, the corresponding (cohomological) Gromov-Witten invariants of degree d are defined by

$$\langle \gamma_1, \ldots, \gamma_n \rangle_d = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(\gamma_1) \wedge \cdots \wedge \mathrm{ev}_n^*(\gamma_n).$$

Let $\mathbb{Z}[q] = \mathbb{Z}[q_\beta : \beta \in \Delta \backslash \Delta_P]$ denote the polynomial ring in variables q_β corresponding to the basis elements of $H_2(X,\mathbb{Z})$. Given any degree $d = \sum_\beta d_\beta [X_{s_\beta}] \in H_2(X,\mathbb{Z})$, we will write $q^d = \Pi_\beta q_\beta^{d_\beta}$. The (small) quantum cohomology ring $\mathrm{QH}(X)$ is a $\mathbb{Z}[q]$-algebra which, as a $\mathbb{Z}[q]$-module, is defined by $\mathrm{QH}^*(X) = H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The product is defined by

$$\gamma_1 \star \gamma_2 = \sum_{w, d \geq 0} \langle \gamma_1, \gamma_2, [X_w] \rangle_d \, q^d [X^w]$$

for $\gamma_1, \gamma_2 \in H^*(X,\mathbb{Z})$. Here we identify any classes $\gamma \in H^*(X,\mathbb{Z})$ with $\gamma \otimes 1 \in \mathrm{QH}(X)$.

In the following, the image of a stable map to X will be called a stable curve in X. Given a stable curve $C \subset X$, we let $[C]$ denote the degree in $H_2(X,\mathbb{Z})$ defined by C. In particular, we set $[C] = 0$ if C is a single point.

2.3. **Distance.** For each $\beta \in \Delta$ we set $Z_\beta = G/P_\beta$, were $P_\beta \subset G$ is the unique maximal parabolic subgroup containing B for which $s_\beta \notin W_{P_\beta}$. The group $H_2(Z_\beta,\mathbb{Z})$ is a free rank one $\mathbb{Z}$-module, generated by $[(Z_\beta)_{s_\beta}]$. To simplify notation we identify $H_2(Z_\beta,\mathbb{Z})$ with $\mathbb{Z}$, by identifying $[(Z_\beta)_{s_\beta}]$ with 1. Let $\pi_\beta : X \to Z_\beta$ denote the projection. Any degree $d \in H_2(X,\mathbb{Z})$ is then given by $d = \sum_\beta d_\beta [X_{s_\beta}]$, where $d_\beta = (\pi_\beta)_*(d)$.

Given $u, v \in W$ let $\mathrm{dist}_\beta(u,v) \in \mathbb{Z}$ denote the smallest degree of a stable curve in Z_β connecting the Schubert varieties $(Z_\beta)^u$ and $(Z_\beta)_v$. This is well defined since the natural numbers are well ordered. Define the *distance* between X^u and X_v to be the class in $H_2(X,\mathbb{Z})$ given by

$$\mathrm{dist}_X(u,v) = \sum_{\beta \in \Delta \backslash \Delta_P} \mathrm{dist}_\beta(u,v)[X_{s_\beta}].$$

The following Theorem is from [6].

Theorem 2.1. *(Buch, Chung, Li, Mihalcea) Let $u, v \in W$ and $d \in H_2(X,\mathbb{Z})$. Then there exists a stable curve of degree d from X^u to X_v if and only if $d \geq \mathrm{dist}_X(u,v)$.*

2.4. **Curve neighborhoods.** Let $d \in H_2(X)$ be an effective degree, and let $\Omega \subset X$ be a closed subvariety. Consider the moduli space of stable maps $\overline{\mathcal{M}}_{0,2}(X,d)$ with evaluation maps $\mathrm{ev}_1, \mathrm{ev}_2$. The *curve neighborhood* of Ω is the subscheme

$$\Gamma_d^X(\Omega) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}\,\Omega) \subset X$$

endowed with a reduced scheme structure. The notion was introduced by Buch, Chaput, Mihalcea and Perrin [5] to help study the quantum K theory ring of cominuscule Grassmannians.

To compute curve neighborhoods we will use the *Hecke product* on the Weyl group W. For a simple reflection s_i the product is defined by

$$w \cdot s_i = \begin{cases} ws_i & \text{if } \ell(ws_i) > \ell(w); \\ w & \text{otherwise} \end{cases}$$

The Hecke product gives W the structure of an associative monoid. Given $u, v \in W$, the product uv is called *reduced* if $\ell(uv) = \ell(u) + \ell(v)$. For any parabolic group P, the Hecke product determines a left action $W \times W/W_P \longrightarrow W/W_P$ defined by

$$u \cdot (wW_P) = (u \cdot w)W_P.$$

Observe from [7] that if Ω is a Schubert variety, then $\Gamma_d^X(\Omega)$ must be a Schubert variety. Let z_d be defined by the condition: $\Gamma_d^X(1.P) = X(z_d W_P)$. The following Proposition describes the curve neighborhood $\Gamma_d^X(X(w))$ as a Schubert variety (see [7]).

Proposition 2.2. *The curve neighborhood of X^w is $\Gamma_d^X(X^w) = X^{w \cdot z_d W_P}$.*

The maximal elements of the set $\{\beta \in R^+ \backslash R_P^+ : \beta^\vee + \Delta^\vee \leq d\}$ are called *maximal roots* of d.

Proposition 2.3. [7, Corollary 4.12] *Let $d \in H_2(X)$ be an effective degree and $w \in W^P$. Then if $\alpha \in R^+ \backslash R_P^+$ is a maximal root of d, then $s_\alpha \cdot z_{d-\alpha^\vee} W_P = z_d W_P$.*

Proposition 2.3 is the key ingrediant that we use to calculate curve neighborhoods in varying combinatorial models through the manuscript.

2.5. **Minimum degree from the point of view of curve neighborhoods.** We begin this subsection with a lemma that states the correspondence between the minimum degree and curve neighborhoods.

Lemma 2.4. *Let $u, v \in W_P$. Let d be the minimal degree such that $X_v \subset \Gamma_d^X(X^u)$ in $X = G/P$. Then d is the smallest power of q in the quantum product $[X^u] \star [X_v]$.*

Proof. Let $u, v \in W_P$. Let d be the minimal degree such that $X_v \subset \Gamma_d^X(X^u)$ in $X = G/P$. Equivalently, $v \leq u \cdot z_d W_P$ in the Bruhat order on W. By definition $d = \mathrm{dist}_X(u, v)$ is the minimum degree of a curve joining two opposite Schubert varieties X^u and X_v. By Theorem 2.1 the degree $\mathrm{dist}_X(u, v)$ is the smallest power of q in the quantum product $[X^u] \star [X_v]$. The result follows. □

For Type A, and the submaximal Grassmanians in Types B and C, we consider four different sets indexing the Schubert subvarieties: the Weyl group, two sets of partitions, and binary strings. For each of these indexing sets, we construct an operator

$$I \to I : \delta \mapsto \delta^d$$

(where I represents any of these indexing sets). The next theorem shows the connection between these constructions, curve neighborhoods, and the minimal quantum degree.

Theorem 2.5. *Let X be one of the following: the Type A (ordinary) Grassmannian, the symplectic Grassmannian, or the odd orthogonal Grassmannian. Let $\delta \in I$ (one of our four indexing sets for Schubert varieties in X). Then*

1. $\Gamma_d(X^\delta) = X^{\delta^d}$
2. *for $\delta_1, \delta_2 \in I$, the smallest power of q in the quantum product $[X^{\delta_1}] \star [X_{\delta_2}]$ is the smallest integer d such that $\delta_2 \leq \delta_1^d$ (in the Bruhat order).*

The proof for part (2) follows from part (1) and Proposition 2.4. The proof for part (1) follows from the fact $\delta^d = \delta \cdot z_d$. We prove this by explicitly identifying the necessary maximal coroots in each Type; for Type A, we do this in Proposition 3.7, for Type C in Proposition 5.17, and Type B in Proposition 5.19.

3. Cominuscule Grassmannians

The purpose of this section is to restate Postnikov's [13, 14] and Fulton and Woodward's [12] minimum degree result in the context of curve neighborhoods and Young diagrams for the Grassmannian. We also prove a Young diagram rule to calculate minimum degrees for Lagrangian Grassmannian, and the maximal odd orthogonal Grassmannian. The curve neighborhoods for the cominscule cases are calculated in [4, Subsection 3.2, Table 1] (See also [5, Lemma 4.2] and [11]). We will discuss the Type A case in its entirety since the proofs of the submaximal cases in Types B and C will follow a similar structure. Then the cominuscule cases in Types B and C will be considered in subsection 3.2.

3.1. **Grassmannian.** Let $\mathrm{Gr} := \mathrm{Gr}(k,n)$ be the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$. The notation in this subsection is strictly for Type A. We begin by defining three indexing sets—permutations, partitions, and 01-words. This is followed by Lemma 3.3 that states the bijections between the index sets.

Definition 3.1. We define the set of partitions $\mathcal{P}(k,n)$.

$$\mathcal{P}(k,n) = \{(\lambda_1 \geq \cdots \geq \lambda_k) : \text{ where } n-k \geq \lambda_1 \text{ and } \lambda_k \geq 0\}.$$

Let P be the maximal parabolic obtained by excluding the reflection s_k. Then the minimal length representatives W^P have the form

$$(w(1) < w(2) < \cdots < w(k) \mid w(k+1) < \cdots < w(n)) \in W^P \subset S_n.$$

Since the last $n-k$ labels are determined from the first, we will identify an element in W^P with the sequence $(w(1) < w(2) < \cdots < w(k))$.

Definition 3.2. Let W_{01} denote the set of 01-words with k ones and $n-k$ zeros;

Lemma 3.3.

(1) *There is a natural bijection between the minimal representatives of W^P and the set W_{01} of 01-words with $n-k$ zeros and k ones. It is given by the following: The element $w \in W^P$ corresponds to the word where the ones appear in the $w(1), w(2), \cdots, w(k)$ positions reading left to right.*

(2) *There is a natural bijection between W_{01} and $\mathcal{P}(k,n)$. It is given as follows: Produce the partition $\lambda \in \mathcal{P}(k,n)$ by reading the 01-word from left to right. Starting at the top-right corner, proceed south-west by moving left for each 0, and down for each 1.*

In Type A, there is a correspondence between partitions in the $k \times (n-k)$ recangle and Grassmannian permutations with descent at k. Given such a partition λ, define

$$u(\lambda)[i] = i + \lambda(k+1-i)$$

for $i = 1, 2, \ldots, k$. For reasons that will become apparent in Type C, we would like permutations to be indexed homologically with partitions indexed cohomologically (i.e. $|\lambda| = \mathrm{codim}(X)$ and $\ell(u) = \dim(X)$ for the same Schubert variety X). For this reason, we would rather use a correspondence between λ and $u_\lambda := u(\lambda^\vee)$.

The following definitions capture the combinatorics of curve neighborhoods in terms of partitions. The first and third definitions correspond to the line neighborhood of the corresponding Schubert variety, and the second and fourth correspond to the degree d curve neighborhood of the corresponding Schubert variety.

Definition 3.4.

(1) Let $\lambda \in \mathcal{P}(k,n)$. Then define $\lambda^1 = (\lambda_2 - 1 \geq \cdots \geq \lambda_k - 1 \geq 0) \in \mathcal{P}(k,n)$ where -1's are replaced by 0.

(2) Define $\lambda^d = (\cdots((\lambda^1)^1)\cdots)^1$ (d-times).
(3) Let $u = (u(1) < u(2) < \cdots < u(k)) \in W^P$. Define u^d in the following way.
  (a) For the case $d = 1$.
    (i) If $u(k) < n$ then define $u^1 := (u(2) < u(3) < \cdots < u(k) < n)$;
    (ii) If

$$u = \left(u(1) < u(2) < \cdots < u(k-j_0-1) < \overbrace{n-j_0}^{u(k-j_0)} < \cdots < \overbrace{n-1}^{u(k-1)} < \overbrace{n}^{u(k)} \right)$$

where $u(k-i) = n-i$ for $0 \leq i \leq j_0$ and $u(k-j_0-1)+1 < n-j_0$ then define

$$u^1 = \left(u(2) < \cdots < \overbrace{u(k-j_0-1)}^{u^1(k-j_0-2)} < \overbrace{n-j_0-1}^{u^1(k-j_0-1)} < \overbrace{n-j_0}^{u^1(n-j_0)} < \cdots < n-1 < n \right).$$

  (b) If $d > 1$ then define u^d recursively by $(u^{d-1})^1$.
(4) Let $\gamma \in W_{01}$ and define γ^1 by: begin by deleting the first 1 occuring in γ, next replace each 10-string with a 01-string, and finally append 1 to the end of the resulting word.
(5) Define γ^d recursively by $\gamma^d = (\gamma^{d-1})^1$.

At this point, we have three combinatorial objects that are related to Schubert varieties: permutations, partitions, and 01-words. We also have bijections between these related objections. We now show that these bijections are actually "dictionaries" which also presurve the combinatorics of curve neighborhoods.

Proposition 3.5. *Suppose $\lambda \in \mathcal{P}(k,n)$, $u \in W^P$, and $\gamma \in W_{01}$ are in bijection with one another. Then λ^d, u^d, and γ^d are in bijection with one another.*

Proof. Observe that λ^1 corresponds to γ^1 by definition. So, it suffices to show that u^1 corresponds to γ^1. Using the correspondence between permutations and 01-words, it is clear that u^1 corresonds to a 01-word obtained from $\gamma(u)$ by interchanging the first 1 and the last 0 occuring in $\gamma(u)$. Thus, the problem is further simplified to: show that γ^1 is constructed by interchanging the first 1 and the last 0 in γ.

Write

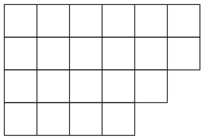
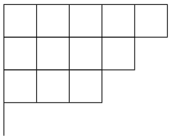
$$\gamma = \mathbf{1}.0.u.1.\mathbf{0}$$

where $\mathbf{0}$ is a string of all zeros, and similar for $\mathbf{1}$. The procedure defining γ^1 produces

$$\gamma^1 = 1.\mathbf{1}.u.0.\mathbf{0} = \mathbf{1}.1.u.0.\mathbf{0}$$

(since replacing all 10-strings with 01-strings results in $u.0$ being replaced by $0.u$). ☐

Example 3.6. Let $n = 12$ and $k = 4$. Consider $u = (3 < 4 < 6 < 8) \in W^P$. Then $u^1 = (4 < 6 < 8 < 12)$, $\gamma = 001101010000$, and $\gamma^1 = 000101010001$, and

$\lambda =$  and $\lambda^1 =$  .

We next specialize Proposition 2.3 to Type A. Let $\{t_i - t_j : 1 \leq i < j \leq n\}$ be the set of positive roots where $\{t_i - t_{i+1} : 1 \leq i \leq n-1\}$ is the set of simple roots.

Proposition 3.7. (1) *The highest coroot is* $t_1 - t_n$ *and it is a maximal element of* $\{\beta \in R^+ \backslash R_P^+ : \beta^\vee + \Delta^\vee \leq 1\}$. *By direct calculation:*

$$z_1 = s_{t_1 - t_n} = s_1 s_2 \dots s_{n-1} \dots s_2 s_1.$$

(2) *Let* $u \in W^P$. *Then*
  (a) $u^1 = u \cdot z_1$;
  (b) $u^d = u \cdot z_1 \cdot \dots \cdot z_1$ *(d-times).*

Proof. The highest root is $t_1 - t_n$ and its coroot is $t_1 - t_n$. Then,

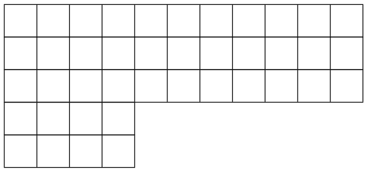
$$t_1 - t_n = (t_1 - t_2) + \cdots + (t_{n-1} - t_n).$$

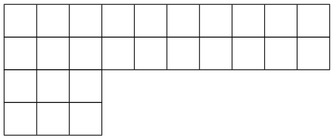
In particular, each coefficient is one. The result follows. $\square$

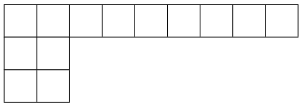
We can now state the Young diagram formula for Type A.

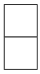
Theorem 3.8. *For any pair of Schubert classes* $[X^\lambda], [X_\mu] \in \mathrm{QH}^*(\mathrm{Gr}(k,n))$ *where* $\lambda, \mu \in \mathcal{P}(k,n)$, *the smallest degree d such that* q^d *appears in* $[X^\lambda] \star [X_\mu]$ *with nonzero coefficient is the smallest integer d such that* $\lambda^d \subset \mu$.

Example 3.9. For $\mathrm{Gr}(5,16)$ consider $\lambda = (11 \geq 11 \geq 11 \geq 4 \geq 4) \in \mathcal{P}(5,16)$ and $\mu = (7,7) \in \mathcal{P}(5,16)$. Then we have that

$\lambda =$  and $\mu =$ 

$\lambda^1 =$ 

$\lambda^2 =$ 

$\lambda^3 =$ .

Thus, 3 is the smallest degree d such that q^d appears in $[X^\lambda] \star [X_\mu]$ with nonzero coefficient.

3.2. The Lagrangian Grassmannian and the maximal odd orthogonal Grassmannian.

We will first define the isotropic Grassmannians in Types B and C. In this subsection we will only consider the cominuscule case for each type. The remainder of the paper will be dedicated to the submaximal cases.

Fix the vector space $\mathbb{C}^{2n}$ with a non-degenerate skew-symmetric bilinear form $(\cdot\,,\cdot)$, and fix a non-negative integer k. The symplectic Grassmannian $\mathrm{IG}(k,2n)$ parameterizes k-dimensional isotropic subspaces of $\mathbb{C}^{2n}$. Similarly, consider the vector space $\mathbb{C}^{2n+1}$ with a non-degenerate symmetric bilinear form. The odd orthogonal Grassmannian $\mathrm{OG}(k,2n+1)$ parameterizes k-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$.

The notation used in this section is only for the cominuscule cases. Let $X \in \{\mathrm{IG}(n,2n), \mathrm{OG}(n,2n+1)\}$. The Schubert varieties are indexed by the set of strict partitions

$$\Lambda := \{(\lambda_1 \geq \cdots \geq \lambda_n) : n \geq \lambda_1, \lambda_k \geq 0, \lambda_i > 0 \implies \lambda_i > \lambda_{i+1}\}.$$

The curve neighborhoods are calculated in [4, Subsection 3.2, Table 1] . We state the curve neighborhoods in the next lemma.

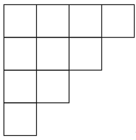
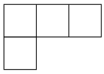
Lemma 3.10. *The following results hold:*

(1) *Let* $X_C^\lambda \subset \mathrm{IG}(n,2n)$ *be a Schubert variety for some* $\lambda_C \in \Lambda$*. Then* $\Gamma_d^{\mathrm{IG}}(X^{\lambda_C}) = X^{\lambda_C^d}$ *where*
$$\lambda_C^d = (\lambda_{d+1} \geq \lambda_{d+1} \geq \cdots \geq \lambda_k \geq 0 \geq \cdots 0).$$
(2) *Let* $X^{\lambda_B} \subset \mathrm{OG}(n,2n)$ *by a Schubert variety for some* $\lambda_B \in \Lambda$*. Then* $\Gamma_d^{\mathrm{OG}}(X^{\lambda_B}) = X^{\lambda_B^d}$ *where*
$$\lambda_B^d = (\lambda_{2d+1} \geq \lambda_{2d+2} \geq \cdots \geq \lambda_k \geq 0 \geq \cdots 0).$$

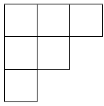
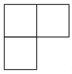
We can now state the Young diagram formula for $\mathrm{IG}(n,2n)$ and $\mathrm{OG}(n,2n+1)$.

Theorem 3.11. *Let* $X \in \{\mathrm{IG}(n,2n), \mathrm{OG}(n,2n+1)\}$*. For any pair of Schubert classes* $[X^\lambda], [X_\mu] \in \mathrm{QH}^*(X)$ *where* $\lambda, \mu \in \Lambda$*, the smallest degree* d *such that* q^d *appears in* $[X^\lambda] \star [X_\mu]$ *with nonzero coefficient is the smallest integer* d *such that* $\lambda^d \subset \mu$*.*

Example 3.12. Consider the case $\mathrm{IG}(5,10)$. Let $\lambda = (4,3,2,1) \in \Lambda$ and $\mu = (3,1) \in \Lambda$. Then we have that

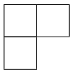
$\lambda =$  and $\mu =$  .

We have that

$\lambda^1 =$  and $\lambda^2 =$  .

So the minimum degree d that appears in the product $[X^\lambda] \star [X_\mu]$ is 2.

Example 3.13. Consider $\mathrm{OG}(5,11)$. Let $\lambda = (4,3,2,1) \in \Lambda$ and $\mu = (3,1) \in \Lambda$. Then we have that

$\lambda^1 =$  .

So the minimum degree d that appears in the product $[X^\lambda] \star [X_\mu]$ is 1.

4. Indexing sets for submaximal Grassmannians in Types B and C

The purpose of this section is to calculate the minimum degree of q for the isotropic Grassmannians $\mathrm{IG}(k,2n)$ (Type C) and $\mathrm{OG}(k,2n+1)$ (Type B). Our results hold for $k < n$ for Type B. We do this in the context of curve neighborhoods analogous to the way we approached the Type A scenario in the previous section. We begin by defining four indexing sets—permutations, two kinds of partitions, and 01-words. The indexing sets for the Schubert varieties of $\mathrm{IG}(k,2n)$ and $\mathrm{OG}(k,2n+1)$ are the same. However, curve neighborhood calculations depend on being in either Type C or Type B. When ambiguity arises, subscripts on elements of indexing sets will be used to indicate whether they are to be considered in Type B or Type C for this section and Section 5. Finally, Lemma 4.5 states the bijections between the index sets.

The associated Weyl group W is the hyperoctahedral group consisting of *signed permutations*, i.e. permutations w of the elements $\{1, \cdots, n, \overline{n}, \cdots, \overline{1}\}$ satisfying $w(\bar{i}) = \overline{w(i)}$ for all $i = 1, \ldots, n$. There is a natural ordering
$$1 < 2 < \ldots < n < \overline{n} < \ldots < \overline{1}.$$

Here $\bar{i} = 2n+1-i$ and $|i| = \min\{i, 2n+1-i\}$.

Let P_k be the maximal parabolic obtained by excluding the reflection s_k. Then the minimal length representatives W^P have the form

$$(w(1) < w(2) < \cdots < w(k) \mid w(k+1) < \cdots < w(n) \leq n) \in W^P$$

if $k < n$ and $(w(1) < w(2) < \cdots < w(n))$ if $k = n$. Since the last $n-k$ labels are determined from the first, we will identify an element in W^P with the sequence

$$(w(1) < w(2) < \cdots < w(k)).$$

The next two definitions include two kinds of partitions. The first are $(n-k)$-strict partitions. The second is a particular subset of $\mathcal{P}(k, 2n)$.

Definition 4.1. Let

$$\Lambda = \{(\lambda_1 \geq \ldots \geq \lambda_k) : 2n-k \geq \lambda_1, \lambda_k \geq 0, \text{ and if } \lambda_j > n-k \text{ then } \lambda_{j+1} < \lambda_j\}$$

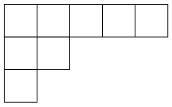
denote the set of $(n-k)$-strict partitions.[1] Let $\ell(\lambda) = \max\{j : \lambda_j > 0\}$ and $\ell_1(\lambda) = \max\{j : \lambda_j > 1\}$.

The next definition states those partitions and a relevant definition. We use the point of view that the 01-words for $\mathrm{IG}(k, 2n)$ and $\mathrm{OG}(k, 2n+1)$ are also 01-words for $\mathrm{Gr}(k, 2n)$.

Definition 4.2. Let $\lambda \in \mathcal{P}(k, n)$ be a partition. This partition's boundary consists of n steps moving either left or down in the south-west direction. Let $D(\lambda)(i) = 0$ if the ith step is left and $D(\lambda)(i) = 1$ if the ith step is down.

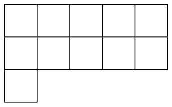
Definition 4.3. Define $\mathcal{P}'(k, 2n) = \{\lambda \in \mathcal{P}(k, 2n) :$ if $D(\lambda)(i) = D(\lambda)(\bar{i})$ then $D(\lambda)(i) = 0\}$. Define W_{01} to be the set of 01-words that correspond to partitions in λ.

Example 4.4. The partition $(5, 2, 1) \in \mathcal{P}'(3, 8)$. The corresponding 01-word is 10001010. Pictorially,



$\in \mathcal{P}'(3, 8)$.

On the contrary, the partition $(5, 5, 1) \notin \mathcal{P}'(3, 8)$. The corresponding 01-word is 11000010. Pictorially,



$\notin \mathcal{P}'(3, 8)$.

The next lemma is dictionary between the different combinatorial models. Part (1) is stated and proved in [9, 10].

Lemma 4.5.

(1) *There is a bijection between Λ, the collection of $(n-k)$-strict partitions, and the set W^P of minimal length representatives given by:*

$$\lambda \mapsto w \quad \text{where } w(j) = 2n+1-k-\lambda_j + \#\{i < j : \lambda_i + \lambda_j \leq 2(n-k) + j - i\},$$
$$w \mapsto \lambda \quad \text{where } \lambda_j = 2n+1-k-w(j) + \#\{i < j : w(i) + w(j) > 2n+1\}.$$

---

[1] It should be noted that while the Bruhat order is compatible with partition inclusion in Type A, the Bruhat orders in Types B and C are not computable by using partition inclusion with k-strict partitions. Indeed, $(2n-k) \leq (1, 1, \cdots, 1)$ in the Bruhat order for $k < n$.

(2) *There is a natural bijection between the minimal length representatives of W^P and the set W_{01}. It is given by the following: The element $w \in W^P$ corresponds to the word where the ones appear in the $w(1), w(2), \cdots, w(k)$ positions reading left to right.*

(3) *The set of W_{01} and $\mathcal{P}'(k, 2n)$ have a canonical bijection. The set $\mathcal{P}'(k, 2n) \subset \mathcal{P}(k, 2n)$ is the set of partitions where $\lambda \in \mathcal{P}'(k, 2n)$ is found by reading the 01-word from left to right. Starting at the top-right corner, proceed south-west by moving left for each 0, and down for each 1.*

5. Curve neighborhood combinatorics

As in Section 3, we provide descriptions of the combinatorics associated to curve neighborhoods. These definitions are spread out among the different types of combinatorics. Definitions 5.1, 5.3 and 5.4 are the k-strict partition point of view in Type C, Type B for $d = 1$ and $k > 1$, and Type B for the other cases, respectively. Definitions 5.6 and 5.9 is the second set of partitions point of view in Types C and B respectively, definitions 5.11 and 5.12 are the permutation point of view in Types C and B respectively, and definition 5.13 is the 01-word perspective (in both Types).

Definition 5.1. Let $\lambda_C \in \Lambda$. Define λ_C^d in the following way:

(1) If $\lambda_1 + \lambda_j > 2(n-k) + j - 1$ for all $2 \leq j \leq k$ then define

$$\lambda_C^1 = (\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_k \geq 0) \in \Lambda;$$

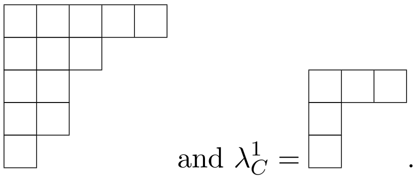
(2) Otherwise, find the smallest j such that $\lambda_1 + \lambda_j \leq 2(n-k) + j - 1$. Define

$$\lambda_C^1 = (\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{j-1} \geq \lambda_j - 1 \geq \cdots \geq \lambda_k - 1 \geq 0) \in \Lambda$$

where -1's are replaced by 0;

(3) Define $\lambda_C^d = (\lambda_C^{d-1})^1$ for $d > 1$.

Example 5.2. Consider the case $n = 8$ and $k = 5$. Let $u_C = (7 < \bar{8} < \bar{5} < \bar{4} < \bar{2}) \in W^P$. Then $u_C^1 = (\bar{8} < \bar{5} < \bar{4} < \bar{2} < \bar{1})$ and



$\lambda_C =$ and $\lambda_C^1 =$.

For curve neighborhood combinatorics in terms $(n-k)$-strict partitions for $\mathrm{OG}(k, 2n+1)$, we first define the $d = 1$ case for $2 \leq k \leq n-1$.

Definition 5.3. Consider the case $2 \leq k \leq n-1$. Let $\lambda_B \in \Lambda$. Define λ_B^1 in the following way:

(1) If $\lambda_1 - \ell(\lambda_B) > 2(n-k)$ then λ_B^1 is defined to be as follows.
  (a) If $\lambda_1 + \lambda_j > 2(n-k) + j - 1$ for all $2 \leq j \leq k$ then define

$$\lambda_B^1 = (\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{\ell(\lambda_B)} \geq \overbrace{1 \geq \cdots \geq 1}^{1+2k+\lambda_1-2n-\ell(\lambda_B)} \geq \overbrace{0 \geq \cdots \geq 0}^{2n-k-\lambda_1}) \in \Lambda;$$

  (b) Otherwise, find the smallest j such that $\lambda_1 + \lambda_j \leq 2(n-k) + j - 1$.
    (i) If $\ell_1(\lambda_B) \geq j$ then

$$\lambda_B^1 = (\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{j-1} \geq \lambda_j - 1 \geq \cdots \geq \lambda_{\ell_1(\lambda_B)} - 1 \geq \overbrace{1 \geq \cdots \geq 1}^{1+2k+\lambda_1-2n-\ell_1(\lambda_B)} \geq \overbrace{0 \geq \cdots \geq 0}^{2n-k-\lambda_1}) \in \Lambda;$$

(ii) If If $\ell_1(\lambda_B) < j$ then

$$\lambda_B^1 = (\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{\ell_1(\lambda_B)} \geq \overbrace{1 \geq \cdots \geq 1}^{1+2k+\lambda_1-2n-\ell_1(\lambda_B)} \geq \overbrace{0 \geq \cdots \geq 0}^{2n-k-\lambda_1}) \in \Lambda;$$

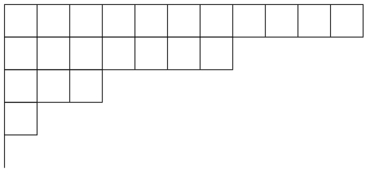
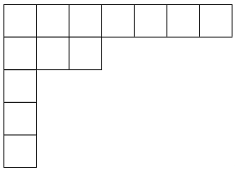
(2) Otherwise, define $\lambda_B^1 := \lambda_C^1$.

Next we define the curve neighborhoods combinatorics in terms of $(n-k)$-strict partitions for the $\mathrm{OG}(1, 2n+1)$ case and for the case when $d > 1$ for $\mathrm{OG}(k, 2n+1)$.

Definition 5.4. Let $\lambda_B \in \Lambda$. Define λ_B^d in the following way:

1. Consider the case $k = 1$. If $\lambda_B = (2n-k)$ then define $\lambda_B^1 = (1)$. If $\lambda_B \neq (2n-k)$ or $d > 1$ then $\lambda_B^d = (0)$.
2. Consider the case $2 \leq k \leq n-1$.
   (a) If $d > 1$ and even then define $\lambda_B^d := \lambda_C^d$.
   (b) If $d > 1$ and odd then define $\lambda_B^d := (\lambda_B^{d-1})^1$.

Example 5.5. Consider the case $n = 8$ and $k = 5$. Let $u_B = (1 < 5 < \bar{8} < \bar{4} < \bar{2}) \in W^P$. Then $u_B^1 = (5 < \bar{8} < \bar{4} < \bar{3} < \bar{2})$ and

$\lambda_B =$  and $\lambda_B^1 =$  .

In this example $\lambda_1 - \ell(\lambda_B) > 2(n-k)$ and $2n - k - \lambda_1 = 0$.

Next we will define the curve neighborhood combinatorics in terms of partitions in $\mathcal{P}'(k, 2n)$. First for Type C.

Definition 5.6. Let $\mu_C \in \mathcal{P}'(k, 2n)$. Then

(1) define $\mu_C^1 = \mu_A^1$. That is,

$$\mu_C^1 := (\mu_2 - 1, \mu_3 - 1, \cdots, \mu_k - 1, 0)$$

where -1's are replaced by 0's.

(2) define $\mu_C^d = (\mu_C^{d-1})^1$ for $d > 1$.

For the Type B combinatorics first recall that the partition $\lambda \in \mathcal{P}(k, n)$ has a boundary consisting of n steps moving either left or down in the south-west direction. We define $D(\lambda)(i)$ to be $D(\lambda)(i) = 0$ if the ith step is left and $D(\lambda)(i) = 1$ if the ith step is down.

Definition 5.7. Let $\lambda \in \mathcal{P}(k, n)$. Let λ^t denote the transpose of λ. We say that λ is *m-wingtip symmetric*[2] if m is the largest nonnegative integer such that

$$D(\lambda)(i) = D(\lambda^t)(i)$$

for $1 \leq i \leq m$.

Example 5.8. Consider $\mu = (10 \geq 8 \geq 3 \geq 1 \geq 0) \in \mathcal{P}'(5, 2 \cdot 8) \subset \mathcal{P}(5, 2 \cdot 8)$. The partition μ is 3-wingtip symmetric. It corresponds to the 01-word 0100100000100101. Here the first and last character, the second and second to last, and the third and third to last character have opposite parody. Pictorially,

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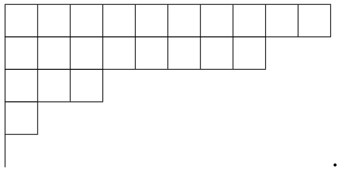
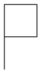
[2] To the best of the authors knowledge this property is unnamed in the literature. The part of the name "wingtip" came from considering Young diagrams as a fixed wing aircraft.

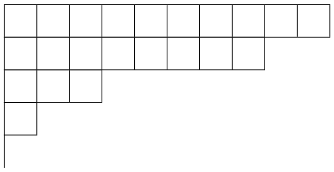
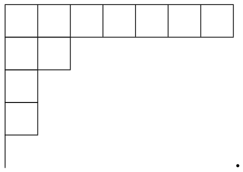
$\mu =$  .

The first 3 bottom left hand boundary edges yield the partition $\gamma =$  and the first 3 top right hand boundary edges yields the partition γ^t.

Definition 5.9. Let $\mu_B \in \mathcal{P}'(k, 2n)$ be m-wingtip symmetric. Then define μ_B^d in the following way.

1. For the case $k = 1$. If $\lambda_B = (2n-k)$ then define $\lambda_B^1 = (1)$. If $\lambda_B \neq (2n-k)$ or $d > 1$ then $\lambda_B^d = (0)$.
2. For the case that $1 < k < 2n$ we have the following subcases.
   (a) If $d = 1$ then μ_B^1 is defined by
      (i) If $D(\mu_B)(\overline{m}) = 1$ and at the end of the ith row then
      $$\mu_B^1 := (\mu_2 - 1 \geq \cdots \geq \mu_{i-1} - 1, \mu_i, \mu_i, \mu_{i+1}, \cdots, \mu_k);$$
      (ii) If $D(\mu_B)(\overline{m}) = 0$, at the bottom of the ith row, and in the jth column then
      $$\mu_B^1 := (\mu_2 - 1 \geq \cdots \geq \mu_i - 1, j, \mu_{i+1}, \cdots, \mu_k).$$
   (b) If $d > 1$ and even define $\mu_B^d := \mu_C^d$;
   (c) If $d > 1$ and odd define $\mu_B^d = (\mu_B^{d-1})^1$.

Example 5.10. Consider $\mu = (10 \geq 8 \geq 3 \geq 1 \geq 0) \in \mathcal{P}'(5, 2 \cdot 8) \subset \mathcal{P}(5, 2 \cdot 8)$. This partition is 3-wingtip symmetric. Pictorially,

$\mu_B =$  and $\mu_B^1 =$  .

Definition 5.11. Let $u_C = (u(1) < u(2) < \cdots < u(k)) \in W^P$. Define u_C^d in the following way.

1. For the case case $d = 1$.
   (a) If $u(k) < \overline{1}$ then define $u_C^1 := (u(2) < u(3) < \cdots < u(k) < \overline{1})$;
   (b) If
   $$u_C = \left(u(1) < u(2) < \cdots < u(k-j_0-1) < \overbrace{\overline{j_0+1}}^{u(k-j_0)} < \cdots < \overbrace{\overline{2}}^{u(k-1)} < \overbrace{\overline{1}}^{u(k)} \right)$$
   where $u(k-i) = \overline{i+1}$ for $0 \leq i \leq j_0$ and $u(k-j_0-1)+1 < \overline{j_0+1}$ then define
   $$u_C^1 = \left(u(2) < \cdots < \overbrace{u(k-j_0-1)}^{u^1(k-j_0-2)} < \overbrace{\overline{j_0}}^{u^1(k-j_0-1)} < \overbrace{\overline{j_0+1}}^{u^1(n-j_0)} < \cdots < \overline{2} < \overline{1} \right).$$
2. If $d > 1$ then define u^d recursively by $(u^{d-1})^1$.

Definition 5.12. Let $u_B \in W^P$.

(1) Consider the case $k = 1$. If $u_B = (1)$ then define $u_B^1 = (\bar{2})$. If $u_B \neq (1)$ or if $d > 1$ then define $u_B^d = (\bar{1})$.

(2) Consider the case $2 \leq k \leq n-1$.
   (a) For the case $d = 1$. Let
$$u_B = (u(1) < u(2) < \cdots < u(k)) .$$
   Let $J = \min\left(\{1, 2, \cdots, n\} \backslash \{|u(1)|, \cdots, |u(k)|\}\right)$. Let $j_0 = \min\left(\{j : u(j) > \bar{J}\} \cup \{k+1\}\right)$.
      (i) If $j_0 = k+1$ then define
$$u_B^1 = \left(u(2) < u(3) < \cdots < u(k) < \bar{J}\right) .$$
      (ii) If $j_0 \leq k$ then define
$$u_B^1 = \left(u(2) < u(3) < \cdots < u(j_0 - 1) < \bar{J} < u(j_0) < \cdots < u(k)\right) .$$
   (b) If $d > 1$ and even then define $u_B^d := u_C^d$.
   (c) If $d > 1$ and odd then define $u_B^d := (u_B^{d-1})^1$.

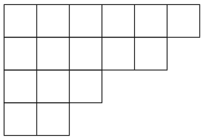
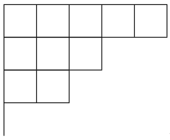
Definition 5.13.

(1) Let $\gamma_C \in W_{01}$ and let $\gamma_C \mapsto u_C \in W^P$. Define $\gamma_C^d \in W_{01}$ to be the 01-word in bijection with u_C^d.
(2) Let $\gamma_B \in W_{01}$ and let $\gamma_B \mapsto u_B \in W^P$. Define $\gamma_B^d \in W_{01}$ to be the 01-word in bijection with u_B^d.

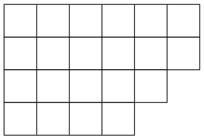
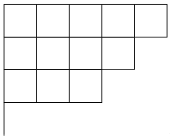
5.1. **Dictionary correspondence for curve neighborhood combinatorics.** The proofs of Propositions 5.14 and 5.16 are reserved until Section 7.

Proposition 5.14. *Suppose $\lambda_C \in \Lambda_C$, $\mu_C \in \mathcal{P}'(k, 2n)$, $u_C \in W^P$, and $\gamma_C \in W_{01}$ are in bijection with one another and $d \geq 0$. Then λ_C^d, μ_C^d, u_C^d, and γ_C^d are in bijection with one another.*

Example 5.15. Let $n = 6$ and $k = 4$. Consider $u_C = (3 < 4 < 6 < \bar{5}) \in W^P$. Then $u_C^1 = (4 < 6 < \bar{5} < \bar{1})$, $\gamma_C = 001101010000$, and $\gamma_C^1 = 000101010001$,

$\lambda_C =$  and $\lambda_C^1 =$  ,

and

$\mu_C =$  and $\mu_C^1 =$  ,

Proposition 5.16. *Let $\lambda_B \in \Lambda_B$, $\mu_C \in \mathcal{P}'(k, 2n)$, $u_B \in W^P$, and $\gamma_B \in W_{01}$ be in bijection with one another and $d \geq 0$. Then λ_B^d, μ_B^d, u_B^d, and γ_B^d are in bijection with one another.*

5.2. **Curve Neighborhoods.** In this subsection we calculate curve neighborhoods for submaximal isotropic Grassmannians in both Types B and C. We begin with Type C.

5.3. **Type C.** We first restate Proposition 2.3 for Type C. Let $\{t_i \pm t_j : 1 \le i \le j \le n\}$ be the set of positive roots where $\{t_i - t_{i+1} : 1 \le i \le n-1\} \cup \{2t_n\}$ is the set of simple roots.

Proposition 5.17. *We restate the recursion for Type C.*

1. *The highest root is $2t_1$ and it is a maximal element of $\{\beta \in R^+ \backslash R_P^+ : \beta^\vee + \Delta^\vee \le 1\}$. By direct calculation: $z_1 = s_{2t_1} = s_1 s_2 \dots s_n \dots s_2 s_1$.*
2. *Let $u_C \in W^P$. Then*
   - (a) *$u_C^1 = u_C \cdot z_1$;*
   - (b) *$u_C^d = u_C \cdot z_1 \cdot \dots \cdot z_1$ (d-times).*

Proof. The key observation is that the highest root is $2t_1$ and its coroot expands a sum of simple coroots as follows:

$$t_1 = (t_1 - t_2) + (t_2 - t_3) + \dots + (t_{n-1} - t_n) + t_n.$$

In particular, each coefficients is one. The result follows. ☐

The next theorem describes the curve neighborhoods of Schubert varieties in $\mathrm{IG}(k, 2n)$.

Theorem 5.18. *Let $\lambda \in \Lambda, \mu \in \mathcal{P}'(k, 2n), u \in W^P, \gamma \in W_{01}$ be in bijection with one another. Consider the Schubert variety $X^\lambda \subset \mathrm{IG}(k, 2n)$. Then*

$$\Gamma_d(X^\lambda) = X^{\lambda^d}; \Gamma_d(X^\mu) = X^{\mu^d}; \Gamma_d(X^u) = X^{u^d}; \Gamma_d(X^\gamma) = X^{\gamma^d}.$$

5.4. **Type B.** As in Type C, we restate Proposition 2.3 for Type B. Recall that we are considering the cases where $k < n$ for $\mathrm{OG}(k, 2n+1)$. The set of positive roots is $\{t_i \pm t_j : 1 \le i < j \le n\} \cup \{t_i : 1 \le i \le n\}$ and the set of simple roots is $\{t_i - t_{i+1} : 1 \le i \le n-1\} \cup \{t_n\}$.

Proposition 5.19. *We restate the recursion for Type B.*

1. *Consider the case $k = 1$.*
   - (a) *The highest root root is $t_1 + t_2$ and it is a maximal element of $\{\beta \in R^+ \backslash R_P^+ : \beta^\vee + \Delta^\vee \le 1\}$. By direct calculation:*
     $$z_1 = s_{t_1+t_2} = (s_2 s_1)(s_3 s_2) \cdots (s_n s_{n-1}) s_n (s_{n-2} s_{n-1}) \cdots (s_1 s_2).$$
   - (b) *Let $u_B \in W^P$. Then*
     - (i) *$u_B^1 = u_B \cdot z_1$;*
     - (ii) *$u_B^d = u_B \cdot z_1 \cdot \dots \cdot z_1$ (d-times);*
2. *Consider the case $2 \le k \le n-1$.*
   - (a) *A maximal element of $\{\beta \in R^+ \backslash R_P^+ : \beta^\vee + \Delta^\vee \le 1\}$ is $t_1 + t_{k+1}$. By direct calculation:*

$$z_1 = s_{t_1+t_{k+1}} = s_1 \cdots s_{k-1} (s_{k+1} s_k) \cdots (s_n s_{n-1}) s_n (s_{n-2} s_{n-1}) \cdots (s_k s_{k+1}) s_{k-1} \cdots s_1;$$

   - (b) *The highest root is $t_1 + t_2$ and it is a maximal element of $\{\beta \in R^+ \backslash R_P^+ : \beta^\vee + \Delta^\vee \le 2\}$. By direct calculation: $z_2 = s_{t_1+t_2}$*
   - (c) *Let $u_B \in W^P$. Then*
     - (i) *$u_B^1 = u_B \cdot z_1$;*
     - (ii) *if d is even then $u_B^d = u_B \cdot z_2 \cdot \dots \cdot z_2$ ($\frac{d}{2}$-times);*
     - (iii) *if d is odd then $u_B^d = u_B \cdot z_2 \cdot \dots \cdot z_2 \cdot z_1$ (z_2 appears $\frac{d-1}{2}$-times).*

Proof. The highest root is $t_1 + t_2$ and its coroot expands as a sum as simple coroots as follows:

$$t_1 + t_2 = (t_1 - t_2) + 2(t_2 - t_3) + \dots + 2(t_{n-1} - t_n) + 2t_n.$$

In particular, $t_1 + t_2$ corresponds to a degree one curve for $k = 1$ and a degree two curve for $1 < k < n$. This establishes z_1 for $k = 1$ and z_2 for $1 < k < n$.

For z_1 in the case $1 < k < n$. We have that

$$t_1 + t_{k+1} = (t_1 - t_2) + \cdots + (t_k - t_{k+1}) + 2(t_{k+1} - t_{k+2}) + \cdots + 2t_n.$$

Every root $\alpha \in R^+$ such that $\alpha > t_1 + t_{k+1}$ takes on the form $\alpha = t_1 + t_j$ where $2 \leq j \leq k$. But the coroot of $t_1 + t_j$ written as sum of simple coroots is

$$t_1+t_j = (t_1-t_2)+\cdots+(t_{j-1}-t_j)+2(t_j-t_{j+1})+\cdots+2(t_k-t_{k+1})+2(t_{k+1}-t_{k+1})+\cdots+2t_n.$$

The result follows. □

The next theorem describes the curve neighborhoods of Schubert varieties in $\mathrm{OG}(k, 2n)$.

Theorem 5.20. *Let $\lambda \in \Lambda, \mu \in \mathcal{P}'(k, 2n), u \in W^P, \gamma \in W_{01}$ be in bijection with one another. Consider the Schubert variety $X^\lambda \subset \mathrm{OG}(k, 2n+1)$. Then*

$$\Gamma_d(X^\lambda) = X^{\lambda^d}; \Gamma_d(X^\mu) = X^{\mu^d}; \Gamma_d(X^u) = X^{u^d}; \Gamma_d(X^\gamma) = X^{\gamma^d}.$$

6. Minimum quantum degree calculation for Types B and C Grassmannians

The purpose of this section is to prove Theorem 1.6. It's important to note for this section that the Weyl group in Types B and C is the hyperoctohedral group on $2n$ elements. So, W_C^P is the set of minimimal length representatives of the hyperoctohedral group. Moreover, the hyperoctohendral group is a subgroup of the symmetric group on $2n$ elements S_{2n}. In this section we will use subscripts to indicate whether an element or an order is to be considered in Type A or Type C. For example, let $\leq_C$ denote the Bruhat order on W_C in Type C and let $\leq_A$ denote the Bruhat order on $W_A := S_{2n}$ in Type A. We begin with a lemma and a proposition regarding the correspondence between the Bruhat order and partition inclusion on partitions in the set $\mathcal{P}'(k, 2n)$. The lemma is proved in [3, Corollary 8.1.9].

Lemma 6.1. *Let $v, w \in W_C \subset W_A$. Then $v \leq_C w$ if and only if $v \leq_A w$. In particular, for any $v, w \in W_C^P$,*

$$v = (v(1) < v(2) < \cdots < v(k)) \leq w = (w(1) < w(2) < \cdots < w(k))$$

if and only if $v_i \leq w_i$ for all $1 \leq i \leq k$.

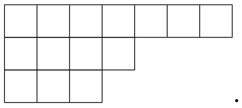
As an immediate application of 6.1 we have the following useful statement regarding partition inclusion and the Bruhat order in Types B and C.

Proposition 6.2. *Let $X \in \{\mathrm{OG}(k, 2n+1), \mathrm{IG}(k, 2n)\}$. We have $X^\mu \subset X^\lambda \subset X$ if and only if $\lambda \subset \mu$ for all $\lambda, \mu \in \mathcal{P}'(k, 2n)$. That is, partition inclusion in $\mathcal{P}'(k, 2n)$ corresponds to the Bruhat order in Types B and C.*

Example 6.3. For an example of the Bruhat order corresponding to partition inclusion consider the case $n = 5$ and $k = 3$. First observe that

$v_C = (1 < 5 < \bar{4}|2 < 3) \in W_C^P$ corresponds to $v_A = (1 < 5 < \bar{4}|2 < 3 < 4 < \bar{5} < \bar{3} < \bar{2} < \bar{1}) \in W_A^P$.

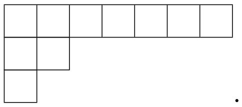
Moreover, both permutations correspond to the 01-word 1000101000 and the partition $\mu \in \mathcal{P}'(3, 10) \subset \mathcal{P}(3, 10)$ given by

$\mu =$  .

Next observe that

$w_C = (1 < \bar{4} < \bar{2}|3 < 5) \in W_C^P$ corresponds to $w_A = (1 < \bar{4} < \bar{2}|2 < 3 < 4 < 5 < \bar{5} < \bar{3} < \bar{1}) \in W_A^P$.

Moreover, both permutations correspond to the 01-word 1000001010 and the partition $\lambda \in \mathcal{P}'(3,10) \subset \mathcal{P}(3,10)$ given by

$\lambda =$  .

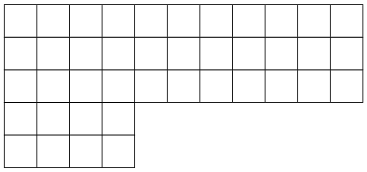
We see that $v_C \leq w_C$, $v_A \leq w_A$, and $\lambda \subset \mu$.

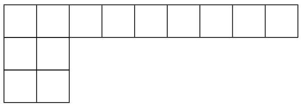
6.1. **Pictorial formulas.** The first theorem of this subsection establishes that the minimum degree that appears in the quantum product of Schubert classes are computed using Young diamgrams.

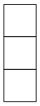
Theorem 6.4. *Let* $X \in \{\mathrm{OG}(k, 2n+1), \mathrm{IG}(k, 2n)\}$. *For any pair of Schubert classes* $[X^\lambda] \star [X_\mu] \in \mathrm{QH}^*(X)$ *where* $\lambda, \mu \in \mathcal{P}'(k, 2n)$*, the smallest degree* d *such that* q^d *appears in* $[X^\lambda] \star [X_\mu]$ *with nonzero coefficient is the smallest integer* d *such that* $\lambda^d \subset \mu$.

Example 6.5. Recall Example 1.7. For $\mathrm{IG}(5, 2\cdot 8)$ consider $\lambda = (11 \geq 11 \geq 11 \geq 4 \geq 4) \in \mathcal{P}'(5, 2\cdot 8)$ and $\mu = (7,7) \in \mathcal{P}'(5, 2\cdot 8)$. We have that 3 is the smallest degree d such that q^d appears in $[X^\lambda] \star [X_\mu]$ with nonzero coefficient.

Example 6.6. For $\mathrm{OG}(5, 2\cdot 8+1)$ consider $\lambda = (11 \geq 11 \geq 11 \geq 4 \geq 4) \in \mathcal{P}'(5, 2\cdot 8)$ and $\mu = (7,7) \in \mathcal{P}'(5, 2\cdot 8)$. Then we have that

$\lambda =$  and $\mu =$ 

$\lambda^2 =$ 

$\lambda^3 =$ 

$\lambda^4 = \emptyset.$

Thus, 4 is the smallest degree d such that q^d appears in $[X^\lambda] \star [X_\mu]$ with nonzero coefficient.

7. Technical Proofs

7.1. **Type C.** We begin with a technical lemma to prove Propositions 5.14. Let $\lambda_C \in \Lambda_C$, $\mu_C \in \mathcal{P}'(k,n)$, $u_C \in W^P$, and $\gamma_C \in W_{01}$ are in bijection with one another and $d \geq 0$. The permutation $u_C^d \in W^P$, the word $\gamma_C^d \in W_{01}$, and the partition $\mu_C^d \in \mathcal{P}'(k,n)$ are in bijection with each other by a direct application of Proposition 5.17 and 3.7. We will complete the proof by showing that $\lambda_C^1 \in \Lambda_C$ and $u_C^1 \in W^P$ are in bijection.

In this section let $|i| = \min\{i, 2n+1-i\}$.

Lemma 7.1. *Let* $u \in W^P$ *where* $u \mapsto \lambda \in \Lambda$. *Suppose that* $u(m) = \overline{\alpha}$ *where* $1 \leq \alpha \leq n$ *and* $\{1, 2, \cdots, \alpha\} \subset \{|u(1)|, |u(2)|, \cdots, |u(k)|\}$. *Then* $\lambda_m = 0$.

Proof. By a direct calculation we have $\#\{i < m : w(i) + \overline{\alpha} > 2n+1\} = k - \alpha$. Then we have that

$$\begin{aligned} \lambda_m &= 2n+1-k-w(m) + \#\{i < m : w(i) + w(j) > 2n+1\} \\ &= \alpha - k + \#\{i < m : w(i) + \overline{\alpha} > 2n+1\} \\ &= 0. \end{aligned}$$

□

Next is the proof of Prop 5.14.

Proof. If $\lambda_1 + \lambda_j > 2(n-k) + j - 1$ for all $2 \leq j \leq k$ then it's straight forward to show that $v^1 \mapsto \lambda^1$.

Otherwise, find the smallest j such that $\lambda_1 + \lambda_j \leq 2(n-k) + j - 1$ and consider

$$\lambda^1 = (\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{j-1} \geq \lambda_j - 1 \geq \cdots \geq \lambda_{k-1} - 1 \geq 0) \in \Lambda$$

where -1's are replaced by 0. Let

$$v = (v(1) < v(2) < \cdots < v(m) < v(m+1) < \cdots < v(k)) \in W^P$$

where $v(k-i) = \overline{1+i}$ for $0 \leq i \leq k-m-1$ and $v(m) < \overline{k-m+1}$ and

$$v^1 = (v(2) < \cdots < v(m) < \overline{1+k-m} < v(m+1) < \cdots < v(k)) \in W^P.$$

It's clear that $v^1(l-1) = v(l) \mapsto \lambda_l$ for $2 \leq l \leq j-1$.

Let l be an integer where $j \leq l \leq m$ and $\lambda_l \geq 1$. Then we have the following

$$\begin{aligned} v^1(l-1) = v(l) &= 2n+1-k-\lambda_l + \#\{i \leq l : \lambda_i + \lambda_l \leq 2(n-k) + l - i\} \\ &= 2n+1-k-\lambda_l + 1 + \#\{2 \leq i \leq l : \lambda_i + \lambda_l \leq 2(n-k) + l - i\} \\ &= 2n+1-k-(\lambda_l - 1) + \#\{2 \leq i \leq l : \lambda_i + \lambda_l \leq 2(n-k) + l - i\} \end{aligned}$$

So $v^1(l-1) \mapsto \lambda_l - 1$ for $j \leq l \leq m$ and $\lambda_l \geq 1$.

Let l be an integer where $j \leq l \leq m$ and $\lambda_l = 0$. Observe the following

$$\begin{aligned} \lambda_l &= 2n+1-k-v(l) + \#\{i < l : v(i) + v(l) > 2n+1\} \\ &\geq 2n+1-k-v(l) + \#\{2 \leq i < l : v(i) + v(l) > 2n+1\} \\ &= 2n+1-k-v^1(l-1) + \#\{i < l : v^1(i) + v^1(l-1) > 2n+1\} \\ &= \lambda^1_{l-1}. \end{aligned}$$

Therefore, $\lambda_l = \lambda^1_{l-1} = 0$ and $v^1(l-1) \mapsto 0 = \lambda_l$ for $j \leq l \leq m$ and $\lambda_l = 0$.

By Lemma 7.1, $\lambda_l = 0$ for $m+1 \leq l \leq k$ and $v^1(m) \mapsto 0$ for $m \leq l \leq k$. So $v^1(l-1) \mapsto 0 = \lambda_l$ for all $m+1 \leq l \leq k$. Observe that $v^1(k) = \overline{1} \mapsto 0 = \lambda^1_k$. The result follows. □

7.2. **Type B.**

Next is the argument of Proposition 5.16. First observe that this is clear for the $k = 1$ case. The cases where $\lambda^d_B = \lambda^d_C$ have already been established by the definitions of u^d_C and u^d_B and λ^d_C and λ^d_B when $d > 1$ and even. We now must consider the case where $d = 1$.

Let $\lambda_B \in \Lambda_B$, $\mu_B \in \mathcal{P}'(k,n)$, $u_B \in W^P$, and $\gamma_B \in W_{01}$ are in bijection. The permutation $u^1_B \in W^P$, the word $\gamma^1_B \in W_{01}$, and the partition $\mu^1_B \in \mathcal{P}'(k,n)$ are in bijection with each other by a direct application of Proposition 5.19. We will complete the proof by showing that

(1) The partition $\lambda^1_B \in \Lambda_C$ and $u^1_C \in W^P$ are in bijection;
(2) The partition $\mu^1_B \in \mathcal{P}'(k,n)$ and $\gamma^1_B \in W_{01}$ are in bijection.

We begin by showing the first listed bijection holds.

7.2.1. *The partition $\lambda_B^1 \in \Lambda_C$ and $u_C^1 \in W^P$ are in bijection.*

Proposition 7.2. *Let $d = 1$ and $\lambda_1 - \ell(\lambda_B) \geq 2(n-k)+1$ then u_B^1 is in bijection with λ_C^1 with all but the last $(2n-k-\lambda_1)$ 0's interchanged with 1's.*

Proof. First note that $\lambda_1 - \ell(\lambda_B) \geq 2(n-k)+1$ is equivalent to $k - \ell(\lambda_B) \geq 2n+1-k-\lambda_1$ Let $\lambda_B \mapsto u_B$. By the bijection we have that $2n+1-k-\lambda_1 = u_B(1)$. It follows that $\{1, 2, \cdots, u_B(1)\} \subset \{|u_B(1)|, |u_B(2)|, \cdots, |u_B(k)|\}$. Thus, $\{1, 2, \cdots, u_B(1)-1\} \subset \{|u_B^1(1)|, |u_B^1(2)|, \cdots, |u_B^1(k)|\}$. In particular, the last $u_B(1)-1 = 2n-k-\lambda_1$ parts of λ_B^1 are zero.

Let m be the largest integer such that $\lambda_m > 0$. By Lemma 7.1 there exists $j \notin \{|u(1)|, |u(2)|, \cdots, |u(k)|\}$ for some $u(m) < j \leq \bar{1}$ and $1 \leq |j| < |u(m)|$. Let J be the largest such j. Next observe that

$$u_C^d := \left(\widehat{u(1)} < u(2) < \cdots < u(m_1) < u(m_2) < \cdots < u(k) < \bar{1}\right)$$

and

$$u_B^1 := \left(\widehat{u(1)} < u(2) < u(3) < \cdots < u(m) < \cdots < u(m_1) < J < u(m_2) < \cdots < \widehat{\overline{u(1)}} \cdots < u(k)\right).$$

Also, $\{1, 2, \cdots, \widehat{u(1)}, \cdots, |J|\} \subset \{|u(1)|, |u(2)|, \cdots, |u(k)|\}$ and $u_B^1(i) = u_C^1(i)$ for $1 \leq i \leq m_1$. The second inequality indicates that $(\lambda_B^1)_i = (\lambda_C^1)_i$ for $1 \leq i \leq m_1$ and nonzero.

By Lemma 7.1 we have that $(\lambda_C^1)_i = 0$ for $m_2 \leq i \leq k$. Consider $(\lambda_B^1)_{i_0}$ for some $i_0, m_2 \leq i_0 \leq k - (2n-k-\lambda_1)$.

By a direct calculation we have $\#\{i < i_0 : w(i) + w(i_0) > 2n+1\} = k - |w(i_0)| + 1$. Then we have that

$$\begin{aligned}
\lambda_{i_0} &= 2n+1-k-w(i_0) + \#\{i < m : w(i) + w(i_0) > 2n+1\} \\
&= |w(i_0)| - k + \#\{i < m : w(i) + w(i_0) > 2n+1\} \\
&= 1.
\end{aligned}$$

The result follows. ☐

Proposition 7.3. *Let $d = 1$ and $\lambda_1 - \ell(\lambda_B) \leq 2(n-k)$. Then u_B^1 is in bijection with λ_C^1.*

Proof. First note that $\lambda_1 - \ell(\lambda_B) \leq 2(n-k)$ is equivalent to $k - \ell(\lambda_B) \leq 2n-k-\lambda_1$. Let $\lambda_B \mapsto u_B$. By the bijection we have that $2n-k-\lambda_1 = u_B(1) - 1$. In particular, by 7.1, $\overline{u_B(1)} \notin \{|u(1)|, |u(2)|, \cdots, |u(k)|\}$. Thus $u_B^1 = u_C^1$. The result follows. ☐

7.2.2. *The partition $\mu_B^1 \in \mathcal{P}'(k, 2n)$ and $\gamma_B^1 \in W_{01}$ are in bijection.*

Lemma 7.4. *The partition $\mu_B^1 \in \mathcal{P}'(k, 2n)$ and $\gamma_B^1 \in W_{01}$ are in bijection.*

Proof. Let $\mu \in \mathcal{P}'(k, n)$ be a m-wingtip symmetric partition. That is, m is the largest nonnegative integer such that $|\gamma(i) - \gamma(2n-i)| = 1$ for $0 \leq i \leq m$. So we have that

$$\gamma_B = \gamma(1) \cdots \gamma(m)\gamma(m+1)\gamma(m+2) \cdots \gamma(2n).$$

Suppose that $\gamma(\overline{m}) = 0$. Then $\gamma(\overline{m+1}) = 0$, otherwise μ_B is not m-wingtip symmetric. So we have that

$$\begin{aligned}
\gamma_B &= \overbrace{0 \cdots 010 \cdots 01 \cdots \gamma(\overline{m+2})0}^{\text{First } 2n-m \text{ characters}}\, 0\gamma(\overline{m-1}) \cdots \gamma(\bar{1}); \\
\gamma_B^1 &= \overbrace{0 \cdots 0\hat{1}0 \cdots 01 \cdots \gamma(\overline{m+2})1}^{\text{First } 2n-m \text{ characters}}\, 0\gamma(\overline{m-1}) \cdots \gamma(\bar{1});.
\end{aligned}$$

Observe that the first $2n-m$ characters correspond to curve neighborhoods in $\mathrm{Gr}(k,2n-m)$ for some k. Thus one shifts the permutation up and deletes a box out of the corresponding rows. Notice that the last rows corresponding to the last m characters of the word will remain the same. The introduction of the character 1 in the $\overline{m+1}$ position forces a row to be of length j (the column before the boundary edge $\gamma(\overline{m})=0$ is in).

Suppose that $\gamma(\overline{m})=1$. Then $\gamma(\overline{m+1})=0$, otherwise μ_B is not m-wingtip symmetric. So we have that

$$\begin{aligned}\gamma_B &= \overbrace{0\cdots010\cdots01\cdots\gamma(\overline{m+2})0}^{\text{First } 2n-m \text{ characters}}1\gamma(\overline{m-1})\cdots\gamma(\overline{1});\\ \gamma_B^1 &= \overbrace{0\cdots0\hat{1}0\cdots01\cdots\gamma(\overline{m+2})1}^{\text{First } 2n-m \text{ characters}}1\gamma(\overline{m-1})\cdots\gamma(\overline{1});.\end{aligned}$$

Observe that the first $2n-m$ characters correspond to curve neighborhoods in $\mathrm{Gr}(k,2n-m)$ for some k. Thus one shifts the permutation up and deletes a box out of the corresponding rows. Notice that the last rows corresponding to the last m characters of the word will remain the same. The introduction of the character 1 in the $\overline{m+1}$ position forces a row to be of length μ_i (the row that boundary edge $\gamma(\overline{m})=1$ is in). $\square$

Proposition 5.16 follows.

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