Some theorems involving Big O

Theorem #1: $O(c\,g(n)) = O(g(n))$ and c is a constant and $c > 0$

Proof:

① $f(n) = O(c\,g(n))$ implies that

there exist constant c_0 and n_0 ($c_0 > 0$, $n > 0$) such that

$$f(n) \le c_0 (c\,g(n)) \quad \text{for all } n \ge n_0$$

Therefore,

$$f(n) \le c_0 \cdot c \cdot g(n)$$

Let $c_1 = c_0 \cdot c > 0$

We have

$$f(n) \le c_1 g(n) \quad \text{for all } n \ge n_0$$

According to definition

② $f(n) = O(g(n))$

Using ① and ②, we conclude

$$O(c\,g(n)) = O(g(n))$$

Theorem #2.

If

$$f_1(n) = O(g_1(n))$$
$$f_2(n) = O(g_2(n))$$

Then

$$f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$$

Proof:

Since $f_1(n) = O(g_1(n))$, we have $c_1 > 0$, $n_1 > 0$

and for all $n \geq n_1$

① $f_1(n) \leq c_1 g_1(n)$

Similarly, we have $c_2 > 0$, $n_2 > 0$ and

for all $n \geq n_2$

② $f_2(n) \leq c_2 g_2(n)$

Therefore, adding ① and ②

$$f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n)$$

for all $n \geq \max(n_1, n_2)$

Let $c = \max(c_1, c_2)$

$$f_1(n) + f_2(n) \leq c(g_1(n) + g_2(n))$$
$$\leq 2c(\max(g_1(n), g_2(n)))$$

Now let $C_0 = 2C$, for all $n \geq \max(n_1, n_2)$

$$f_1(n) + f_2(n) \leq C_0 \cdot \max(g_1(n), g_2(n))$$

Therefore, we can have

$$f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$$

Theorem #3.

If $f(n)$ is polynomial of degree x.

then $f(n) = O(n^x)$

Theorem #4

If

$$f_1(n) = O(g_1(n))$$
$$f_2(n) = O(g_2(n))$$

Then

$$f_1(n) * f_2(n) = O(g_1(n) * g_2(n))$$