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Numerical Integration of a Linear Barotropic Model Using Three Methods of Treating Meteorological and Gravitational Modes Separately

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Summary

The analytical solution of a linear barotropic model is derived, including details of the quasi-geostrophic initialization procedure. The prognostic equations are integrated using three different methods of treating the meteorological and gravitational modes separately. These are a semi-Eulerian, semi-implicit (EI) technique, a semi-Lagrangian, semi-implicit (LI) procedure, and a split-explicit (SE) method. The stability criteria and phase speeds are derived for each of the three techniques.

The following theoretical conclusions are derived. Of course, in actual numerical integrations particularly those using more complex models, the results are not so unequivocal.

The stability of the EI procedure is governed by the CFL criterion for the meteorological mode. Gravity waves have no effect on the timestep but move more slowly than the analytical waves. The LI method is unconditionally stable with respect to both meteorological and gravitational modes. There is thus no timestep restriction. However, the gravity waves have the same reduced phase speed as in the EI technique. The SE procedure has CFL timestep criteria for both the meteorological and gravitational calculations. However, its gravity wave phase speeds are relatively accurate. Moreover, it is the only one of the three methods that handles the nearly-compensating pressure gradient and Coriolis forces together. From the point of computational efficiency, the LI technique is probably the best.

1. Introduction

It is well-known that the hydrostatic primitive equations admit as solutions high-speed gravity waves as well as slower-moving meteorological waves (e.g., Haltiner and Williams, 1980, pp. 40–42). If the equations are integrated explicitly in time, the fast-moving gravity waves require a short time step in order to satisfy the Courant-Friedrichs-Lewy (CFL) stability criterion. The time step is much less than that needed to keep time truncation error acceptably small. This results in an excessively long computational time to produce a prognosis.

To overcome this problem, several methods have been designed to treat the gravity and meteorological modes separately. Three of these will be investigated in this paper. One is a semi-Eulerian, semi-implicit procedure (adapted from Kwizak and Robert, 1971), hereafter referred to as the EI method. A second is a semi-Lagrangian, semi-implicit technique (Robert, 1981), which will be designated as LI. A third is a split-explicit method (Marchuk, 1974), which will be called SE.

Research is continuing on these procedures. For example, Kar et al. (1994) describe an EI technique which yields a locally one-dimensional method of solving a two-dimensional Helmholtz equation. Purser and Leslie (1994) present a LI technique in which forward trajectories, rather than the usual backward trajectories, are calculated. Rivest et al. (1994) describe problems and remedies associated with orographic forcing in a LI model. Gallée and Schayes (1994) use a form of the SE method in which advection is calculated using a semi-Lagrangian scheme.

In the rest of this paper, the EI, LI and SE methods are applied to a linearized barotropic model. This model is described in Section 2. The advantages of such a simple model are that it yields an analytical solution and permits an easy comparison of the three procedures. Details of the derivation of the initialization technique of Phillips (1960) have been added, which are not in the original paper. The numerical procedures of the EI, LI and SE procedures applied to the model are described in Sections 3, 4 and 5, respectively. Stability criteria and phase speeds are derived for each technique. The advantages and disadvantages of the three methods as applied to the simple model are summarized in Section 6.

Some of the results presented here are probably familiar individually. However, a contribution of this paper is throught to be the synthesis of various findings in such a way as to facilitate understanding and comparing them. The derivations of the stability criteria and phase speeds of the EI and LI methods do not appear to be widely published in the literature.

2. A Linear Barotropic Model

The shallow-water equations (e.g., Haltiner and Williams, 1980, p. 54) may be written, omitting metric terms due to the earth's sphericity,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial \Phi}{\partial x}-fv=0, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial \Phi}{\partial y}+fu=0, \tag{2.2}$$

$$\frac{\partial \Phi}{\partial t}+u\frac{\partial \Phi}{\partial x}+v\frac{\partial \Phi}{\partial y}+\Phi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0. \tag{2.3}$$

Here the x-axis points east, the y-axis points north, u and v are the x and y velocity components, Φ is the geopotential, and f is the Coriolis parameter.

Consider a basic state with a uniform zonal current U in geostrophic balance. That is,

$$U=-\frac{1}{f}\frac{\partial \hat{\Phi}}{\partial y}, \tag{2.4}$$

where $\hat{\Phi}(y)$ is the geopotential of the basic state. Superimpose a perturbation independent of y. The total velocity components and geopotential may be expressed as

$$u(x,t)=U+u'(x,t), \tag{2.5}$$

$$v(x,t)=v'(x,t), \tag{2.6}$$

$$\Phi(x,y,t)=\hat{\Phi}(y)+\phi'(x,t), \tag{2.7}$$

where the primes denote perturbations. From Eqs. (2.4) and (2.7),

$$v\frac{\partial \Phi}{\partial y}=-fUv. \tag{2.8}$$

Substituting Eqs. (2.5)–(2.8) in (2.1)–(2.3), assuming the Coriolis parameter has a constant value f_0, replacing Φ in the last term of Eq. (2.3) by a constant Φ_0, replacing $u(\partial/\partial x)$ by $U(\partial/\partial x)$, and dropping the primes yields

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial x}-f_0v=0, \tag{2.9}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+f_0u=0, \tag{2.10}$$

$$\frac{\partial \phi}{\partial t}+U\frac{\partial \phi}{\partial x}+\Phi_0\frac{\partial u}{\partial x}-f_0Uv=0. \tag{2.11}$$

where u, v and ϕ now denote perturbation quantities.

An analytical solution to Eqs. (2.9)–(2.11) may be obtained following the method of Kurihara (1965). Some details have been added which are not in Kurihara's paper. Assume travelling waves of the type

$$u=\tilde{u}\exp[i\nu(x-ct)], \tag{2.12}$$

$$v=\tilde{v}\exp(i\nu(x-ct)], \tag{2.13}$$

$$\phi=\tilde{\phi}\exp[i\nu(x-ct)], \tag{2.14}$$

where $\tilde{u}$, $\tilde{v}$ and $\tilde{\phi}$ are complex amplitudes, $\nu=2\pi/L$ is the wave number, L is the wavelength, and c is the phase speed. From Eqs. (2.12)–(2.14),

$$\frac{\partial \alpha}{\partial t}=-c\frac{\partial \alpha}{\partial x}, \tag{2.15a}$$

$$\frac{\partial \alpha}{\partial x}=i\nu\alpha, \tag{2.15b}$$

where $\alpha=u$, v or ϕ. Applying Eqs. (2.15) to (2.9)–(2.11) yields

$$i(U-c)\nu u-f_0v+i\nu\phi=0, \tag{2.16}$$

$$f_0 u + i(U-c)\nu v = 0, \tag{2.17}$$

$$i\nu\phi_0 u - f_0 U v + i(U-c)\nu\phi = 0. \tag{2.18}$$

For a non-trivial solution, the determinant of the coefficients of Eqs. (2.16)–(2.18) must vanish. This gives the following cubic equation for the phase speed:

$$(U-c)^3 - \Phi_0(U-c) + \frac{f_0^2}{\nu^2}c = 0. \tag{2.19}$$

Kurihara (1965, Eqs. (3.3)) gives the following roots of Eq. (2.19):

$$c_1 = U + 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\varepsilon}{3}+\frac{4\pi}{3}\right), \tag{2.20}$$

$$c_2 = U + 2\sqrt{-\frac{a}{3}}\cos\frac{\varepsilon}{3}, \tag{2.21}$$

$$c_3 = U + 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\varepsilon}{3}+\frac{2\pi}{3}\right), \tag{2.22}$$

where

$$\varepsilon = \tan^{-1}\left(-\frac{4a^3}{27b^3}-1\right)^{1/2},$$

$$a = -\frac{f_0^2}{\nu^2} - \Phi_0,$$

$$b = -\frac{f_0^2 U}{\nu^2}.$$

Equation (2.20) is the phase speed for the meteorological or Rossby mode. Equations (2.21) and (2.22) are the phase speeds for the two gravitational modes.

Approximate solutions for the phase speeds may be obtained by writing Eq. (2.19) as follows:

$$\left(\frac{U-c}{c_0}\right)^2 - \frac{f_0^2}{\nu^2 c_0^2(1-U/c)} - 1 = 0, \tag{2.23}$$

where

$$c_0^2 = \Phi_0. \tag{2.24}$$

For the Rossby mode, one would expect $(U-c) \ll c_0$ so the first term in Eq. (2.23) may be omitted. This results in

$$c_1 = \frac{U}{1+f_0^2/(\nu^2 c_0^2)}. \tag{2.25}$$

For the gravitational modes, $U/c \ll 1$ so this term will be dropped from the denominator of the second term in Eq. (2.23). This yields

$$c_2 = U + \sqrt{c_0^2 + f_0^2/\nu^2}, \tag{2.26}$$

$$c_3 = U - \sqrt{c_0^2 + f_0^2/\nu^2}. \tag{2.27}$$

Equations (2.25)–(2.27) will be used when discussing initialization. If effects of rotation are neglected by setting $f_0 = 0$, Eqs. (2.25)–(2.27) further simplify to

$$c_1 = U, \tag{2.28}$$

$$c_2 = U + c_0, \tag{2.29}$$

$$c_3 = U - c_0. \tag{2.30}$$

The complete analytical solution to Eqs. (2.9)–(2.11) will now be presented. Let

$$\phi_j = \tilde{\phi}_j \exp[i\nu(x - c_j t)], \quad j = 1,2,3, \tag{2.31}$$

where $\tilde{\phi}_j$ is the geopotential amplitude of the jth mode. Equations (2.16)–(2.18) must hold fox each of the three modes. Eliminating v between Eqs. (2.16) and (2.17) results in

$$u_j = \frac{(U-c_j)\nu^2}{f_0^2 - (U-c_j)^2\nu^2}\phi_j, \quad j = 1,2,3. \tag{2.32}$$

Substituting Eq. (2.32) into (2.17) gives

$$v_j = \frac{i\nu f_0}{f_0^2 - (U-c_j)^2\nu^2}\phi_j, \quad j = 1,2,3. \tag{2.33}$$

Equations (2.31)–(2.33) are Eqs. (3.2) of Kurihara (1965). The complete solution is

$$u = \sum_{j=1}^{3} u_j, \tag{2.34a}$$

$$v = \sum_{j=1}^{3} v_j, \tag{2.34b}$$

$$\phi = \sum_{j=1}^{3} \phi_j. \tag{2.34c}$$

The amplitudes $\tilde{\phi}_j$ in Eqs. (2.31)–(2.34) are arbitrary. Nevertheless, the two gravitational modes may be removed from the initial conditions by following the procedure of Phillips (1960), also discussed by Haltiner and Williams (1980, pp. 45–47). However, these references omit many of the details of the derivation, which is non-trivial. These details will now be given. Set $t = 0$ in Eqs.

(2.31)–(2.34):

$$u(x,0)=\exp(i\nu x)\sum_{j=1}^{3}\tilde{\phi}_j\left[\frac{\nu^2(U-c_j)}{f_0^2-(U-c_j)^2\nu^2}\right]$$
$$=u_0\exp(i\nu x), \tag{2.35}$$

$$v(x,0)=\exp(i\nu x)\sum_{j=1}^{3}\tilde{\phi}_j\left[\frac{i\nu f_0}{f_0^2-(U-c_j)^2\nu^2}\right]$$
$$=v_0\exp(i\nu x), \tag{2.36}$$

$$\phi(x,0)=\exp(i\nu x)\sum_{j=1}^{3}\tilde{\phi}_j=\phi_0\exp(i\nu x), \tag{2.37}$$

where

$$u_0=\sum_{j=1}^{3}\tilde{u}_j,$$

$$v_0=\sum_{j=1}^{3}\tilde{v}_j$$

$$\phi_0=\sum_{j=1}^{3}\tilde{\phi}_j$$

are the total amplitudes. Let

$$\mu=c_0^2+f_0^2/\nu^2. \tag{2.38}$$

Assume the phase speeds are given by Eqs. (2.25)–(2.27) so that

$$U-c_1=Uf_0^2\nu^{-2}\mu^{-1}, \tag{2.39a}$$

$$U-c_2=-\mu^{1/2}, \tag{2.39b}$$

$$U-c_3=\mu^{1/2}. \tag{2.39c}$$

Substituting Eqs. (2.39a), (2.39b) and (2.39c) in (2.35)–(2.37) and dividing by $\exp(i\nu x)$ yields the following three equations for $\tilde{\phi}_1, \tilde{\phi}_2$ and $\tilde{\phi}_3$:

$$\frac{Uf_0^2\mu^{-1}}{f_0^2-U^2f_0^4\nu^{-2}\mu^{-2}}\tilde{\phi}_1-\frac{\nu^2\mu^{1/2}}{f_0^2-\mu\nu^2}\tilde{\phi}_2+\frac{\nu^2\mu^{1/2}}{f_0^2-\mu\nu^2}\tilde{\phi}_3$$
$$=u_0, \tag{2.40}$$

$$\frac{i\nu f}{f_0^2-U^2f_0^4\nu^{-2}\mu^{-2}}\tilde{\phi}_1+\frac{i\nu f_0}{f_0^2-\mu\nu^2}\tilde{\phi}_2+\frac{i\nu f_0}{f_0^2-\mu\nu^2}\tilde{\phi}_3$$
$$=v_0, \tag{2.41}$$

$$\tilde{\phi}_1+\tilde{\phi}_2+\tilde{\phi}_3=\phi_0. \tag{2.42}$$

Equations (2.40)–(2.42) may be solved for $\tilde{\phi}_1, \tilde{\phi}_2$ and $\tilde{\phi}_3$ in terms of u_0, v_0 and ϕ_0. After much algebra, the results are

$$\tilde{\phi}_1=\left(\frac{1-U^2f_0^2\nu^{-2}\mu^{-2}}{U^2f_0^4\nu^{-4}\mu^{-2}-\mu}\right)f_0^2\nu^{-2}(if_0^{-2}\nu c_0^2v_0-\phi_0), \tag{2.43}$$

$$\tilde{\phi}_2=\frac{c_0^2}{2\mu^{1/2}}u_0+c_0^2\left[\frac{i\nu^{-1}f_0(1+U\mu^{-1/2})v_0+\phi_0}{2(Uf_0^2\nu^{-2}\mu^{-1/2}+\mu)}\right], \tag{2.44}$$

$$\tilde{\phi}_3=-\frac{c_0^2}{2\mu^{1/2}}u_0-c_0^2\left[\frac{i\nu^{-1}f_0(1-U\mu^{-1/2})v_0+\phi_0}{2(Uf_0^2\nu^{-2}\mu^{-1/2}-\mu)}\right]. \tag{2.45}$$

Consider the first factor of Eq. (2.43). In the numerator,

$$U^2f_0^2\nu^{-2}\mu^{-2}=(Uf_0^2\nu^{-2}\mu^{-1})U\mu^{-1}$$
$$=(U-c_1)U\mu^{-1}\ll 1,$$

where Eq. (2.39a) has been employed in the second equality. Similarly, in the denominator of Eq. (2.43),

$$U^2f_0^4\nu^{-4}\mu^{-2}=(U-c_1)^2\ll\mu.$$

Thus Eq. (2.43) may be simplified to

$$\tilde{\phi}_1=f_0^2\nu^{-2}\mu^{-1}(-if_0^{-1}\nu c_0^2v_0+\phi_0). \tag{2.46}$$

Consider now the denominators of the second terms in Eqs. (2.44) and (2.45).

$$Uf_0^2\nu^{-2}\mu^{-1/2}=(U-c_1)\mu^{1/2}\ll\mu.$$

Therefore Eqs. (2.44) and (2.45) may be simplified to

$$\tilde{\phi}_2=\frac{c_0^2}{2\mu^{1/2}}u_0+c_0^2\left[\frac{i\nu^{-1}f_0(1+U\mu^{-1/2})v_0+\phi_0}{2\mu}\right], \tag{2.47}$$

$$\tilde{\phi}_3=-\frac{c_0^2}{2\mu^{1/2}}u_0+c_0^2\left[\frac{i\nu^{-1}f_0(1-U\mu^{-1/2})v_0+\phi_0}{2\mu}\right]. \tag{2.48}$$

It is desirable to specify initial conditions free of gravity waves. Suppose initial conditions are given by

$$\frac{\partial^2 u}{\partial x^2}-\frac{f_0^2}{\Phi_0}u=\frac{U}{\Phi_0}\frac{\partial^2\phi}{\partial x^2}, \tag{2.49}$$

$$v=\frac{1}{f_0}\frac{\partial\phi}{\partial x}. \tag{2.50}$$

Equation (2.49) is obtained by substituting Eq.

(2.50) (the geostrophic relation) into Eqs. (2.10) and (2.11) and eliminating time derivatives from the resulting two equations. Since $\partial\alpha/\partial y = 0$, where $\alpha = u$ or v, u makes no contribution to the vorticity

$$\zeta = \frac{\partial v}{\partial x}, \tag{2.51}$$

and v makes no contribution to the divergence

$$\delta = \frac{\partial u}{\partial x}. \tag{2.52}$$

Thus u may be interpreted as the divergent part of the wind and v as the rotational part. The rotational part is calculated geostrophically by Eq. (2.50) while the divergent part is evaluated from the quasi-geostrophic Eq. (2.49).

In Eqs. (2.49) and (2.50), substitute for u, v and ϕ using the right hand sides of the second equalities in Eqs. (2.35)–(2.37). This gives, with the use of Eqs. (2.15b), (2.24) and (2.38),

$$u_0 = U\mu^{-1}\phi_0, \tag{2.53}$$

$$v_0 = ivf_0^{-1}\phi_0. \tag{2.54}$$

Substituting Eqs. (2.53) and (2.54) in Eqs. (2.46)–(2.48) results in

$$\tilde{\phi}_1 = \phi_0, \tag{2.55}$$

$$\tilde{\phi}_2 = \tilde{\phi}_3 = 0. \tag{2.56}$$

Thus, initializing by means of Eqs. (2.49) and (2.50) eliminates gravity waves from the initial conditions. Note that using the geostrophic relation for the divergent part of the wind (i.e. $u = 0$) instead of Eq. (2.49) will not yield Eq. (2.56), and hence will not completely remove gravity waves. The purpose of Phillips' (1960) technique is similar to that of linear normal mode initialization.

3. A Semi-Eulerian Semi-Implicit (EI) Method

A grid-point version of the EI method following Kwizak and Robert (1971), will be applied to Eqs. (2.9)–(2.11). Let F be any function and define the following differencing and averaging operations:

$$F_s = \frac{F(s+\Delta s) - F(s-\Delta s)}{2\Delta s}, \tag{3.1}$$

$$\bar{F}^s = \frac{F(s+\Delta s) + F(s-\Delta s)}{2}, \tag{3.2}$$

where s is an independent variable (x or t). In the EI technique, one writes Eqs. (2.9)–(2.11) as

$$u_t + \bar{\phi}_x^t = -Uu_x + f_0 v = a(t), \tag{3.3}$$

$$v_t = -Uv_x - f_0 u = b(t), \tag{3.4}$$

$$\phi_t + \Phi_0 \bar{u}_x^t = -U\phi_x + f_0 U v = c(t). \tag{3.5}$$

Thus the pressure gradient force in Eq. (3.3) and the divergence in Eq. (3.5) are regarded as gravitational terms and treated implicitly (time-averaged). The remaining terms in Eqs. (3.3)–(3.5) are considered meteorological terms and calculated explicitly. A possible source of problems in the EI method is the fact that the nearly compensating pressure gradient and Coriolis forces are treated separately. The acceleration is usually a small imbalance between these two forces. In the LI technique (discussed in Section 4), the two forces are handled together in one equation but separately in a second equation. In the SE procedure (Section 5), both forces are treated as gravitational terms.

Rearrange Eqs. (3.3)–(3.5) to give

$$u(t+\Delta t) + \Delta t\phi_x(t+\Delta t) = u(t-\Delta t) - \Delta t\phi_x(t-\Delta t) + 2\Delta t a(t) = A(t, t-\Delta t), \tag{3.6}$$

$$v(t+\Delta t) = v(t-\Delta t) + 2\Delta t b(t) = B(t, t-\Delta t), \tag{3.7}$$

$$\phi(t+\Delta t) + \Phi_0 \Delta t u_x(t+\Delta t) = \phi(t-\Delta t) - \Phi_0 \Delta t u_x(t-\Delta t) + 2\Delta t c(t) = C(t, t-\Delta t). \tag{3.8}$$

The value of $v(t+\Delta t)$ can be evaluated directly from Eq. (3.7). Eliminate u from Eqs. (3.6) and (3.8) by taking the x-difference of Eq. (3.6) and substituting in Eq. (3.8):

$$\phi(t+\Delta t) - \Phi_0(\Delta t)^2\phi_{xx}(t+\Delta t) = C - \Phi_0 \Delta t A_x \tag{3.9}$$

Equation (3.9) is a finite difference elliptic equation in $\phi(t+\Delta t)$. When it is solved for $\phi(t+\Delta t)$, this is substituted in Eq. (3.6) to yield $u(t+\Delta t)$.

The stability and phase speeds will now be derived. By analogy to the analytical solutions (2.12)–(2.14), assume numerical solutions of the type

$$u(n\Delta t, m\Delta x) = \tilde{u}\exp[i(\alpha n\Delta t + vm\Delta x)], \tag{3.10}$$

$$v(n\Delta t, m\Delta x) = \tilde{v}\exp[i(\alpha n\Delta t + vm\Delta x)], \tag{3.11}$$

$$\phi(n\Delta t, m\Delta x) = \tilde{\phi}\exp[i(\alpha n\Delta t + vm\Delta x)], \tag{3.12}$$

where m and n are integers, $n\Delta t$ and $m\Delta x$ are discrete values of t and x, $\alpha = 2\pi/\tau$ is the angular frequency, and τ is the period. Substitute Eqs.

Table 1. *Values Assigned to Various Parameters*

U	$25\,\mathrm{m\,s^{-1}}$
Δx	50 km
L	1000 km
Φ_0	$5.46 \times 10^4\,\mathrm{m^2\,s^{-2}}$
f_0	$10^{-4}\,\mathrm{s^{-1}}$

(3.10)–(3.12) in Eqs. (3.3)–(3.5) to give

$$i\left(\frac{\sin\alpha\Delta t}{\Delta t}+\frac{U\sin\nu\Delta x}{\Delta x}\right)u-f_0v$$
$$+i\frac{\sin\nu\Delta x\cos\alpha\,\Delta t}{\Delta x}\phi=0, \quad (3.13)$$

$$f_0u+i\left(\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}\right)v=0, \quad (3.14)$$

$$i\Phi_0\frac{\sin\nu\,\Delta x\cos\alpha\,\Delta t}{\Delta x}u-f_0Uv$$
$$+i\left(\frac{\sin\alpha\,t\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}\right)\phi=0. \quad (3.15)$$

Equations (3.13)–(3.15) are the analogues of Eqs. (2.16)–(2.18).

As in the analytical case, for a non-trivial solution, the determinant of the coefficients of Eqs. (3.13)–(3.15) must vanish. This results in

$$\left(\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}\right)^3-\Phi_0\left(\frac{\sin\nu\,\Delta x\cos\alpha\,\Delta t}{\Delta x}\right)^2$$
$$\times\left(\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}\right)$$
$$+f_0^2\left[\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}(1-\cos\alpha\,\Delta t)\right]=0. \quad (3.16)$$

Equation (3.16) is the analogue of Eq. (2.19). The magnitudes of the terms in Eq. (3.16) may be computed using the values in Table 1. Assuming $\Delta t/\tau \leqslant 0.1$, the last term of Eq. (3.16) (involving f_0^2) is four orders of magnitude smaller than the others and may be omitted. Thus rotational effects are negligible in Eq. (3.16). The corresponding analytical phase speeds are given by Eqs. (2.28)–(2.30). The simplified Eq. (3.16) yields three equations:

$$\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}=0, \quad (3.17)$$

$$\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}+\Phi_0^{1/2}\frac{\sin\nu\,x\cos\alpha\,\Delta t}{\Delta x}=0, \quad (3.18)$$

$$\frac{\sin\alpha\,\Delta t}{\Delta t}+\frac{U\sin\nu\,\Delta x}{\Delta x}-\Phi_0^{1/2}\frac{\sin\nu\,\Delta x\cos\alpha\,\Delta x}{\Delta x}=0. \quad (3.19)$$

For computational stability, it is necessary that α be real in each of Eqs. (3.17)–(3.19). For the meteorological mode, write Eq. (3.17) as

$$\sin\alpha\Delta t=-\frac{U\Delta t\sin\nu\,\Delta x}{\Delta x}. \quad (3.20)$$

To ensure that α is real, one requires that $|\sin\alpha\,\Delta t| \leqslant 1$. From Eq. (3.20), since $|\sin\nu\,\Delta x|\leqslant 1$, a sufficient condition for stability is

$$\Delta t\leqslant\frac{\Delta x}{U}. \quad (3.21)$$

For the values given in Table 1, (3.21) gives $\Delta t \leqslant 2000$s. Kwizak and Robert (1971) and Robert (1979) refer to the necessity of satisfying the CFL criterion (3.21).

Equations (3.18) and (3.19) may be used to give the stability criteria for the gravitational models. In Eq. (3.18) define

$$X=\frac{1}{\Delta t}, \quad (3.22)$$

$$Y_1=\Phi_0^{1/2}\frac{\sin\nu\,\Delta x}{\Delta x}, \quad (3.23)$$

$$\sin\beta_1=\frac{Y_1}{\sqrt{X^2+Y_1^2}}, \quad (3.24)$$

$$\cos\beta_1=\frac{X}{\sqrt{X^2+Y_1^2}}. \quad (3.25)$$

Substituting Eqs. (3.22) and (3.23) in Eq. (3.18) gives

$$X\sin\alpha\,\Delta t+Y_1\cos\alpha\,\Delta t=-\frac{U\sin\nu\,\Delta x}{\Delta x}. \quad (3.26)$$

However, from Eqs. (3.24) and (3.25),

$$X=\sqrt{X^2+Y_1^2}\cos\beta_1,$$
$$Y_1=\sqrt{X^2+Y_1^2}\sin\beta_1.$$

Therefore Eq. (3.26) may be written

$$\sin(\alpha\,\Delta t + \beta_1) = -\frac{U \sin \nu\,\Delta x}{\Delta x\sqrt{X^2 + Y_1^2}}. \tag{3.27}$$

Since Δt and β_1 are real, to guarantee that α is real, it is necessary that $(\alpha\Delta t - \beta_1)$ be real. A sufficient condition is that the square of the right side of Eq. (3.27) not exceed unity. This gives, with the substitution of Eqs. (3.22) and (3.23), the stability criterion

$$-(\Delta t)^2 \sin^2 \nu\,\Delta x(\Phi_0 - U^2) \leqslant (\Delta x)^2. \tag{3.28}$$

Since $U^2 \ll \Phi_0$ for any realistic values of U and Φ_0 (cf. Table 1), (3.28) is satisfied for any value of Δt.

As for the other gravitational mode, in Eq. (3.19) define

$$Y_2 = -\frac{\Phi_0^{1/2} \sin \nu\,\Delta x}{\Delta x}, \tag{3.29}$$

$$\sin \beta_2 = \frac{Y_2}{\sqrt{X^2 + Y_2^2}}, \tag{3.30}$$

$$\cos \beta_2 = \frac{X}{\sqrt{X^2 + Y_2^2}}. \tag{3.31}$$

Employing an argument analogous to that used for Eq. (3.18) yields

$$\sin(\alpha\Delta t + \beta_2) = -\frac{U \sin \nu\,\Delta x}{\Delta x\sqrt{X^2 + Y_2^2}}, \tag{3.32}$$

which is similar to Eq. (3.27) and which also gives (3.28). Thus the EI method is theoretically unconditionally stable with respect to gravity waves. Its stability is governed by the CFL criterion (3.21) for the meteorologica, model.

Using Eq. (3.20), the phase speed of the meteorological mode in the computational solution is given by

$$c_1^* = -\frac{\alpha}{\nu} = \frac{1}{\nu\Delta t}\sin^{-1}\left(\frac{U\Delta t}{\Delta x}\sin \nu\,\Delta x\right). \tag{3.33}$$

Employing the values in Table 1 and setting $U\Delta t/\Delta x = 1$ (see (3.21)) in Eq. (3.33), one obtains $c_1^* = U = 25\,\mathrm{m\,s^{-1}}$, which agrees exactly with the analytical phase speed given by Eq. (2.28). The more accurate analytical phase speed given by Eq. (2.20) is $c_1 = 24.885\,\mathrm{m\,s^{-1}}$.

Gravity waves may be removed from the initial conditions using the procedure described in the last part of Section 2 (Eqs. (2.35)–(2.56)). However, if any gravity waves exist in the numerical solution, it is instructive to examine their phase speeds. From Eqs. (3.27) and (3.32),

$$c_2^* = \frac{1}{\nu\Delta t}\left[\sin^{-1}\left(\frac{U \sin \nu\,\Delta x}{\Delta x\sqrt{X^2 + Y_1^2}}\right) + \beta_1\right], \tag{3.34}$$

$$c_3^* = \frac{1}{\nu\,\Delta t}\left[\sin^{-1}\left(\frac{U \sin \nu\,\Delta x}{\Delta x\sqrt{X^2 + Y_2^2}}\right) + \beta_2\right], \tag{3.35}$$

where

$$\beta_1 = \sin^{-1}\left(\frac{Y_1}{\sqrt{X^2 + Y_1^2}}\right), \tag{3.36}$$

$$\beta_2 = \sin^{-1}\left(\frac{Y_2}{\sqrt{X^2 + Y_2^2}}\right). \tag{3.37}$$

Substitute $\Delta t = \Delta x/U$ (see (3.21)) and Eqs. (3.22), (3.23), (3.29), (3.36) and (3.37) into Eqs. (3.34) and (3.35) to give

$$c_2^* = \frac{U}{\nu\Delta x}\left[\sin^{-1}\left(\frac{U \sin \nu\,\Delta x}{\sqrt{U^2 + \Phi_0 \sin^2 \nu\,\Delta x}}\right) + \sin^{-1}\left(\frac{\Phi_0^{1/2} \sin \nu\,\Delta x}{\sqrt{U^2 + \Phi_0 \sin^2 \nu\,\Delta x}}\right)\right], \tag{3.38}$$

$$c_3^* = \frac{U}{\nu\Delta x}\left[\sin^{-1}\left(\frac{U \sin \nu\,\Delta x}{\sqrt{U^2 + \Phi_0 \sin^3 \nu\,\Delta x}}\right) - \sin^{-1}\left(\frac{\Phi_0^{1/2} \sin \nu\,\Delta x}{\sqrt{U^2 + \Phi_0 \sin^2 \nu\,\Delta x}}\right)\right]. \tag{3.39}$$

Using the values in Table 1, Eqs. (3.38) and (3.39) yield $c_2^* = 106\,\mathrm{m\,s^{-1}}$ and $c_3^* = -90\,\mathrm{m\,s^{-1}}$. These are considerably smaller than the analytical values given by Eqs. (2.29) and (2.30) which are $c_2 = 259\,\mathrm{m\,s^{-1}}$ and $c_3 = -209\,\mathrm{m\,s^{-1}}$. Mesinger and Arakawa (1976, p. 58) note the retardation of gravity waves.

4. A Semi-Lagrangian Semi-Implicit (LI) Method

Following Robert (1981), replace Eqs. (2.9) and (2.10) with the vorticity and divergence equations

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + f_0\delta = 0, \tag{4.1}$$

$$\frac{\partial \delta}{\partial t} + U\frac{\partial \delta}{\partial x} + \frac{\partial^2 \phi}{\partial x^2} - f_0\zeta = 0. \tag{4.2}$$

Write Eq. (2.11) as

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + \Phi_0\delta - f_0 U v = 0. \tag{4.3}$$

In the LI method, write Eq. (4.1) as

$$\frac{\zeta(x,t+\Delta t)-\zeta(x-2U\Delta t,t-\Delta t)}{2\Delta t} + f_0\delta(x-U\Delta t,t)=0. \tag{4.4}$$

The reason for employing the vorticity equation is that ζ is quasi-conservative. That is, the pattern of ζ is approximately advected with speed U without change in shape, since the last term in Eqs. (4.1) and (4.4) (the divergence term) is normally small. The values at $(x-2U\Delta t,\ t-2\Delta t)$ and $(x-U\Delta t, t-\Delta t)$ in Eq. (4.4) are found by interpolation. In Eq. (4.4), one needs to know δ. Evaluate Eqs. (4.2) and (4.3) semi-implicitly:

$$\delta_t + \bar{\phi}^t_{xx} = -U\delta_x + f_0\zeta = a(t), \tag{4.5}$$

$$\phi_t + \Phi_0\bar{\delta}^t = -U\phi_x + f_0 Uv = b(t). \tag{4.6}$$

Note that in Eq. (4.5), the terms due to the pressure gradient force (ϕ_{xx}) and the Coriolis force ($f_0\zeta$) are handled differently. This was also done in the EI procedure (Section 3). However, in Eq. (4.4), the pressure gradient force has been eliminated and the term due to the Coriolis force (last term) is treated as an explicit forcing term. Eq. (4.6) is identical to Eq. (3.5).

Write Eqs. (4.5) and (4.6) as

$$\delta(t+\Delta t) + \Delta t\phi_{xx}(t+\Delta t) = \delta(t-\Delta t) + \Delta t\phi_{xx}(t-\Delta t) + 2\Delta t a(t) = A(t,t-\Delta t), \tag{4.7}$$

$$\phi(t+\Delta t) + \Phi_0\Delta t\delta(t+\Delta t) = \phi(t-\Delta t) + \Phi_0\Delta t\delta(t-\Delta t) + 2\Delta t b(t) = B(t,t-\Delta t). \tag{4.8}$$

Eliminate $\delta(t+\Delta t)$ from these two equations to yield

$$\phi(t+\Delta t) - \Phi_0(\Delta t)^2\phi_{xx}(t+\Delta t) = B - \Phi_0\Delta t A. \tag{4.9}$$

Note that the left side of Eq. (4.9) is identical to that of Eq. (3.9). Solve Eq. (4.9) for $\phi(t+\Delta t)$ and substitute in Eqs. (4.7) or (4.8) to obtain $\phi(t+\Delta t)$. New velocities are found from

$$u = \chi_x, \tag{4.10}$$

$$v = \psi_x, \tag{4.11}$$

where

$$\chi_{xx} = \delta, \tag{4.12}$$

$$\psi_{xx} = \zeta \tag{4.13}$$

are solved to give the new velocity potential χ and stream function ψ. In the simple linear model examined here, only Eqs. (4.11) and (4.13) need be evaluated every timestep, since v is required in the right side of Eq. (4.6). Equations (4.10) and (4.12) are not needed until the end of the prognosis.

To examine the stability of the LI method, it is convenient to express ζ and δ in Eqs. (4.4)–(4.6) in terms of u and v. From Eqs. (2.51), (2.52), (3.10) and (3.11),

$$\zeta = v_x = i\frac{v\sin\nu\Delta x}{\Delta x}, \tag{4.14}$$

$$\delta = u_x = i\frac{u\sin\nu\Delta x}{\Delta x}. \tag{4.15}$$

Equation (4.4) may then be written

$$f_0 e^{-i\nu U\Delta t}u + \frac{(e^{i\alpha\Delta t} - e^{-i\alpha\Delta t}e^{-2i\nu U\Delta t})}{2\Delta t}v = 0. \tag{4.16}$$

However,

$$e^{i\alpha\Delta t} - e^{-i\alpha\Delta t}e^{-2i\nu\Delta t} = e^{-i\nu U\Delta t}[e^{i(\alpha+\nu U)\Delta t} - e^{-i(\alpha+\nu U)\Delta t}] = 2ie^{-i\nu U\Delta t}\sin(\alpha+\nu U)\Delta t.$$

Hence, Eq. (4.16) may be simplified to

$$f_0 u + \frac{i\sin(\alpha+\nu U)\Delta t}{\Delta t}v = 0. \tag{4.17}$$

Equations (4.5) and (4.6) may be written

$$\frac{\sin\nu\Delta x}{\Delta x}\left(\frac{\sin\alpha\Delta t}{\Delta t} + \frac{U\sin\nu\Delta x}{\Delta x}\right)u + \frac{if_0\sin\nu\Delta x}{\Delta x}v + \frac{\sin^2\nu\Delta x}{(\Delta x)^2}(\cos\alpha\Delta t)\phi = 0, \tag{4.18}$$

$$i\Phi_0\frac{\sin\nu\Delta x}{\Delta x}(\cos\alpha\Delta t)u - f_0 Uv + i\left(\frac{\sin\alpha\Delta t}{\Delta t} + \frac{U\sin\nu\Delta x}{\Delta x}\right)\phi = 0. \tag{4.19}$$

Equations (4.17)–(4.19) are the analogues of Eqs. (2.16)–(2.18) and (3.13)–(3.15). Equations (4.19) and (3.15) are identical. Setting the determinant of the coefficients of Eqs. (4.17)–(4.19) to zero yields

$$\frac{\sin\nu\Delta x}{\Delta x}\frac{\sin(\alpha+\nu U)\Delta t}{\Delta t}\left[\left(\frac{\sin\alpha\Delta t}{\Delta t} + \frac{U\sin\nu\Delta x}{\Delta x}\right)^2 - \Phi_0\frac{\sin^2\nu\Delta x}{(\Delta x)^2}\cos^2\alpha\Delta t\right]$$

$$+f_0^2\left[-\frac{\sin\nu\Delta x}{\Delta x}\left(\frac{\sin\alpha\Delta t}{\Delta t}+\frac{U\sin\nu\Delta x}{\Delta x}\right)+\frac{U\sin^2\nu\Delta x}{(\Delta x)^2}\cos\alpha\Delta t\right]=0. \quad (4.20)$$

Equation (4.20) is the analogue of Eqs. (2.19) and (3.16).

Omitting the last term (involving f_0^2) in Eq. (4.20) as before, the simplified equation yields the following three relationships:

$$\sin(\alpha+\nu U)\Delta t=0, \quad (4.21)$$

$$\frac{\sin\alpha\Delta t}{\Delta t}+\frac{U\sin\nu\Delta x}{\Delta x}+\Phi_0^{1/2}\frac{\sin\nu\Delta x\cos\alpha\Delta t}{\Delta x}=0, \quad (4.22)$$

$$\frac{\sin\alpha\Delta t}{\Delta t}+\frac{U\sin\nu\Delta x}{\Delta x}-\Phi_0^{1/2}\frac{\sin\nu\Delta x\cos\alpha\Delta t}{\Delta x}=0. \quad (4.23)$$

Since ν and U are real, Eq. (4.21) will theoretically guarantee that α is real no matter what value is assigned to Δt. Equations (4.22) and (4.23) are identical to Eqs. (3.18) and (3.19) and hence will always yield a real α. Therefore the LI method is theoretically unconditionally stable with respect to both the meteorological mode and the two gravitational modes.

The phase speed of the meteorological mode is given by Eq. (4.21) as

$$c_1^{**}=-\frac{\alpha}{\nu}=U, \quad (4.24)$$

which is the same as the analytical speed given by Eq. (2.28). Equations (4.22) and (4.23) yield the phase speeds of the gravitational modes

$$c_2^{**}=c_2^*, \quad (4.25)$$

$$c_3^{**}=c_3^*, \quad (4.26)$$

where c_2^* and c_3^*, the phase speeds in the EI method, are given by Eqs. (3.34) and (3.35).

5. A Split-Explicit (SE) Method

Following Marchuk (1974), split Eqs. (2.9)–(2.11) into meteorological and gravitational terms:

$$\frac{\partial u}{\partial t}=-U\frac{\partial u}{\partial x}, \quad (5.1)$$

$$\frac{\partial u}{\partial t}=-\frac{\partial\phi}{\partial x}+f_0 v, \quad (5.2)$$

$$\frac{\partial v}{\partial t}=-U\frac{\partial v}{\partial x}, \quad (5.3)$$

$$\frac{\partial v}{\partial t}=-f_0 u, \quad (5.4)$$

$$\frac{\partial\phi}{\partial t}=-U\frac{\partial\phi}{\partial x}+f_0 U v, \quad (5.5)$$

$$\frac{\partial\phi}{\partial t}=-\Phi_0\frac{\partial u}{\partial x}. \quad (5.6)$$

Equations (5.1), (5.3) and (5.5) are meteorological equations and include the advective terms. Recall from Eq. (2.8) that the last term of Eq. (5.5) is the y-component of geopotential advection. Equations (5.2), (5.4) and (5.6) are gravitational equations and account for the remaining terms. The Coriolis forces in Eqs. (5.2) and (5.4) are regarded as gravitational terms so as not to separate them from the pressure gradient forces. There is no y-component of the pressure gradient force in the gravitational Eq. (5.4).

Let Δt and δt be the timesteps for the meteorological and gravitational equations, respectively. Let $\hat{u}$, $\hat{v}$ and $\hat{\phi}$ be values at $(t+\Delta t)$ computed from finite difference versions of the meteorological Eqs. (5.1), (5.3) and (5.5). That is,

$$\hat{u}=u(t-\Delta t)-2\Delta t U u_x(t), \quad (5.7)$$

$$\hat{v}=v(t-\Delta t)-2\Delta t U v_x(t), \quad (5.8)$$

$$\hat{\phi}=\phi(t-\Delta t)+2\Delta t[-U\phi_x(t)+f_0 U v(t)]. \quad (5.9)$$

Then integrate finite difference versions of the gravitational Eqs. (5.2), (5.4) and (5.6) from t to $(t+\Delta t)$ in steps of δt, starting with $\hat{u}$, $\hat{v}$ and $\hat{\phi}$ at time t. Thus

$$u(t+\delta t)=u(t-\delta t)+2\delta t[-\phi_x(t)+f_0 v(t)], \quad (5.10)$$

$$v(t+\delta t)=v(t-\delta t)-2\delta t f_0 u(t), \quad (5.11)$$

$$\phi(t+\delta t)=\phi(t-\delta t)-2\delta t\Phi_0 u_x(t). \quad (5.12)$$

To determine the stability criteria and phase speeds, apply Eqs. (3.10)–(3.12) to Eqs. (5.7)–(5.12). Eqs. (5.7)–(5.9) give

$$i\left(\frac{\sin\alpha\Delta t}{\Delta t}+U\frac{\sin\nu\Delta x}{\Delta x}\right)u=0, \quad (5.13)$$

$$i\left(\frac{\sin\alpha\,\Delta t}{\Delta t}+U\frac{\sin\nu\,\Delta x}{\Delta x}\right)v=0, \tag{5.14}$$

$$-f_0Uv+i\left(\frac{\sin\alpha\,\Delta t}{\Delta t}+U\frac{\sin\nu\,\Delta x}{\Delta x}\phi=0.\right. \tag{5.15}$$

Setting the determinant of the coefficients of Eqs. (5.13)–(5.15) to zero yields Eq. (3.20), the result for the EI method. Thus the stability criterion and phase speed of the meteorological mode are given by (3.21) and (3.33).

Equations (5.10)–(5.12) yield

$$i\frac{\sin\alpha\,\delta t}{\delta t}u-f_0v+\frac{i\sin\nu\,\Delta x}{\Delta x}\phi=0, \tag{5.16}$$

$$f_0u+i\frac{\sin\alpha\,\delta t}{\delta t}v=0, \tag{5.17}$$

$$i\,\Phi_0\frac{\sin\nu\,\Delta x}{\Delta x}u+i\frac{\sin\alpha\,\delta t}{\delta t}\phi=0. \tag{5.18}$$

Setting the determinant of the coefficients of Eqs. (5.16)–(5.18) to zero yields, after dividing out the common factor $\sin\alpha\,\delta t/\delta t$,

$$\left(\frac{\sin\alpha\,\delta t}{\delta t}\right)^2-\Phi_0\frac{\sin^2\nu\,\Delta x}{(\Delta x)^2}+f_0^2=0. \tag{5.19}$$

The term f_0^2 will be omitted as before, although in this case its retention creates no problem. The resulting simplified Eq. (5.21) yields

$$\sin\alpha\,\delta t=\mp\frac{\Phi_0^{1/2}\,\delta t}{\Delta x}\sin\nu\,\Delta x. \tag{5.20}$$

For α to be real, a sufficient condition in Eq. (5.20) is

$$\delta t\leqslant\frac{\Delta x}{\Phi_0^{1/2}}. \tag{5.21}$$

The phase speeds are given by Eq. (5.20) as

$$c_2^{***}=-\frac{\alpha}{\nu}=\frac{1}{\nu\delta t}\sin^{-1}\left(\frac{\Phi_0^{1/2}\delta t}{\Delta x}\sin\nu\,\Delta x\right), \tag{5.22}$$

$$c_3^{***}=-c_2^{***}. \tag{5.23}$$

Comparing Eqs. (5.22) and (5.23) with the analytical expressions (2.29) and (2.30), it is seen that the basic current U is not included in Eqs. (5.22) and (5.23). Using the values in Table 1, and the upper limit to the timestep in Eq. (5.21), one obtains from Eqs. (5.22) and (5.23) the speeds $c_2^{***}=\Phi_0^{1/2}=234\,\mathrm{m\,s^{-1}}$ and $c_3^{***}=-234\,\mathrm{m\,s^{-1}}$. The analytical values from Eqs. (2.28) and (2.30) are $c_2=259\,\mathrm{m\,s^{-1}}$ and $c_3=-209\,\mathrm{m\,s^{-1}}$.

A variation of the SE method is the calculation of the advection terms by a Lagrangian technique with the remaining terms being computed in an explicit, Eulerian manner (Krishnamurti, 1962; Gallée and Schayes, 1994).

6. Concluding Remarks

The following theoretical findings are obtained. It should be noted, however, that the results are determined analytically using a very simple model. In actual numerical integrations using more realistic models, the results are not so clear-cut.

The stability of the EI method is determined by the CFL criterion for the meteorological mode. High speed gravity waves do not limit the timestep but the implicit part of the EI procedure results in slow gravity wave speeds. As Mesinger and Arakawa (1976, p. 58) point out, this may affect the geostrophic adjustment process. Nevertheless, Collins (1980) does not find any significant difference in adjustment time and accuracy of the final adjusted state between explicit and semi-implicit calculations. The LI technique is unconditionally stable with respect to both the meteorological and gravitational modes so there is no timestep restriction. However, it has the same reduced gravity wave speeds as the EI procedure. The SE method has CFL timestep limitations for both the meteorological and gravity wave calculations. However, the gravity waves have more accurate phase speeds than in the EI and LI techniques. Moreover, it is the only procedure of the three that treats the nearly compensating pressure gradient and Coriolis forces together.

If one is not concerned with the slow movement of the gravity waves, from the point of view of computation efficiency, the LI method is probably preferable, since it can use the largest timesteps of the three procedures. A glance at the literature suggests that much of the current work is on LI techniques.

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