Exploiting Symmetry in Pascal’s Triangle Modulo a Prime

The goal of this lab is to use patterns in Pascal’s triangle modulo a prime to derive conjectures for some binomial identities that hold modulo a prime $p$. A binomial identity is a formula involving binomial coefficients. In your investigation, you will be interpreting a pattern in Pascal’s triangle modulo $p$, and it will be useful to make additional identifications among the group elements. Your investigation will involve an application of the (action of the) dihedral symmetry group of the triangle.

**Exercise 1** Generate the first $p - 1$ rows of Pascal’s triangle modulo $p$ for various primes $p$ and classify the symmetry group of the resulting triangles. Repeat the above, but this time color every element $a$ the same as its inverse $-a$.

The groups you classified in exercise (1) act by symmetries on the set of entries, or cells, of the triangle — each of the symmetries of the triangle maps the set of cells back onto itself. The orbit of any cell has at most six elements — that is a given cell can be sent to up to as many as 6 distinct cells (including itself) under the group action. Cells in the same orbit will have the same color, and hence the binomial coefficients they represent in $\mathbb{Z}_p$ are equal (up to a sign in the case where you identified $a$ with $-a$).

Recall the full symmetry group of the triangle, $D$, is generated by reflections in any two median lines. To understand the relationship between the binomial coefficients in a given orbit, it suffices to look at the orbits of cells under these two motions only. The image in Figure 1 shows one way to coordinatize the cells in the triangle (here we have taken $p = 13$).

![Figure 1: Each pair $(s,t)$ $(s,t \geq 0$ and $s + t \leq p - 1$) corresponds to the cell $\binom{p-1-s}{t}$. The cell labeled with an “a” has coordinates $(s,t) = (8,3)$ and represents the residue class of $\binom{4}{3} \, (\text{mod } 13)$. The cells labeled “a” through “f” comprise an orbit and the binomial coefficients they represent are all congruent either to 4 or $-4$ modulo 13.](image)

**Exercise 2** The vertical symmetry (which holds without the identification of the signs) gives a binomial identity. Find it. Does this symmetry depend on the sign identification?

**Exercise 3** Prove the identity you conjectured in exercise (2).

**Exercise 4** Referring to Figure 1, note that the symmetry across the line $s = t$ does depend on the signs. Look at Pascal’s triangle modulo $p$ and decide when the symmetry across the line $s = t$ depends on the sign identification. Write your result as a conjecture for another binomial identity.

**Exercise 5** Use your conjecture from exercises (2) and (4) together with the fact that these two reflections generate the group $D$ to conjecture three more (non-trivial) binomial identities.

**Exercise 6** Prove the identities you conjectured in exercises (4) and (5).