Lecture 5:
Spatial Algorithms

GEOG 419: Advanced GIS
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Spatial Analysis Algorithms
► Basis of much of GIS analysis today
► Manipulation of map coordinates
► Based on Euclidean coordinate geometry
► http://astronomy.swin.edu.au/~pbourke/geometry/

A matter of analytical cartography
► Theoretical and mathematical background behind cartography
► Seeks to find how geographic properties of space can be used in analysis, modeling, and prediction
► AC consists of the basic mathematical algorithms principles of cartography that survive independently of a particular technology
AC Algorithms

- Algorithm - special method for solving a problem stated as a formula or a set of sequential instructions
- Church's Theorem - if a problem can be stated as a series of sequential instructions, then it can be automated
- Cartographic transformational algorithms are the nuts and bolts from which GIS are constructed

Transformational View of Cartography

- Types of data/types of maps
- Map scale
- Dimensional
- Symbolization
- Generalization
- Data model

Transformations

- Goal is to express a transformation as an explicit math operation so that \( 0 \rightarrow 1 \) in a fully described way and an inverse transformation \( 1 \rightarrow 2 = 0 \)
- Invertable transformations
  - Special subset, allows for prediction of error, spatial modeling
- Point-to-point transformations - very central, more than any other
- Can have multi-step transformations
Dimensional Transformations

- Coordinate transformations
  - Projections
- Geometric transformations
  - Measurements from coordinates
- Affine transformations
  - Rotation, translation, and scaling
- Statistical space transformations

Planar map transformations

- Distance between two points
  - Figured out 2,400 years ago by Pythagoras
    \[ d_{out} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
  - Is this invertable? Only if we have three points
- Length of a line
  \[ \text{length} = \sum_{i=1}^{npts} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \]

Planar map transformations II

- Weighted average point
- Cities \((x, y, P)\)  \(P = \text{population}\)
  \[ x = \frac{\sum_{i=1}^{npts} P_i x_i}{\sum_{i=1}^{npts} P_i} \quad \quad y = \frac{\sum_{i=1}^{npts} P_i y_i}{\sum_{i=1}^{npts} P_i} \]
The equations of the lines are:

\[
P_a = P_1 + u_a (P_2 - P_1)
\]

\[
P_b = P_3 + u_b (P_4 - P_3)
\]

Solving for the point where \( P_a = P_b \) gives the following two equations with two unknowns \( u_a \) and \( u_b \):

\[
x_1 + u_a (x_2 - x_1) = x_3 + u_b (x_4 - x_3)
\]

\[
y_1 + u_a (y_2 - y_1) = y_3 + u_b (y_4 - y_3)
\]

Solving for \( u_a \) and \( u_b \), we get:

\[
u_a = \frac{(x_4 - x_3)(y_1 - y_3) - (y_4 - y_3)(x_1 - x_3)}{(x_4 - y_3)(x_2 - x_1) - (y_4 - x_3)(y_2 - y_1)}
\]

\[
u_b = \frac{(x_2 - x_1)(y_1 - y_3) - (y_2 - y_1)(x_1 - x_3)}{(x_4 - y_3)(x_2 - x_1) - (y_4 - x_3)(y_2 - y_1)}
\]

Substituting either into the equation for a line gives us the intersection point:

\[
x = x_1 + u_a (x_2 - x_1)
\]

\[
y = y_1 + u_a (y_2 - y_1)
\]
Intersection of two lines

- If the denominator above is 0, the lines are parallel.
- If the denominator and the numerator above is 0, the lines are coincident.
- In the computer, we have a real problem with 0.
  - Actual zero almost never happens.
  - Means we must check within limits of coordinate precision.
- Solution is not elegant – test every combination of line segments.
  - Bounding Box heuristic.

Distance between a point and a line

- Equation for the line
  \[ P = P_1 + u(P_2 - P_1) \]
- The shortest distance from \( P_3 \) to \( P \) is a perpendicular line. This means the dot product of the line and the perpendicular is 0.
  \[ (P_3 - P) \cdot (P_2 - P_1) = 0 \]
- Substitute for the equation of the line
  \[ ((P_3 - P_1) - u(P_2 - P_1)) \cdot (P_2 - P_1) = 0 \]
Distance between point and line

- Solve for \( u \)
  \[ u = \frac{(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)}{(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2})} \]
- Find the intersection point
  \[ x = x_1 + u(x_2 - x_1) \]
  \[ y = y_1 + u(y_2 - y_1) \]
- Distance is length of line between \( P_3 \) and intersection point

Area of a polygon

- Project lines from each vertex to some outside perpendicular. Each area under the line is a triangle and a rectangle. Sum the areas as you move around the polygon. Outside areas get subtracted out.

**Example**

\[ A = \frac{1}{2} \sum_{i=1}^{n} (x_{i+1}y_i - x_iy_{i+1}) \]