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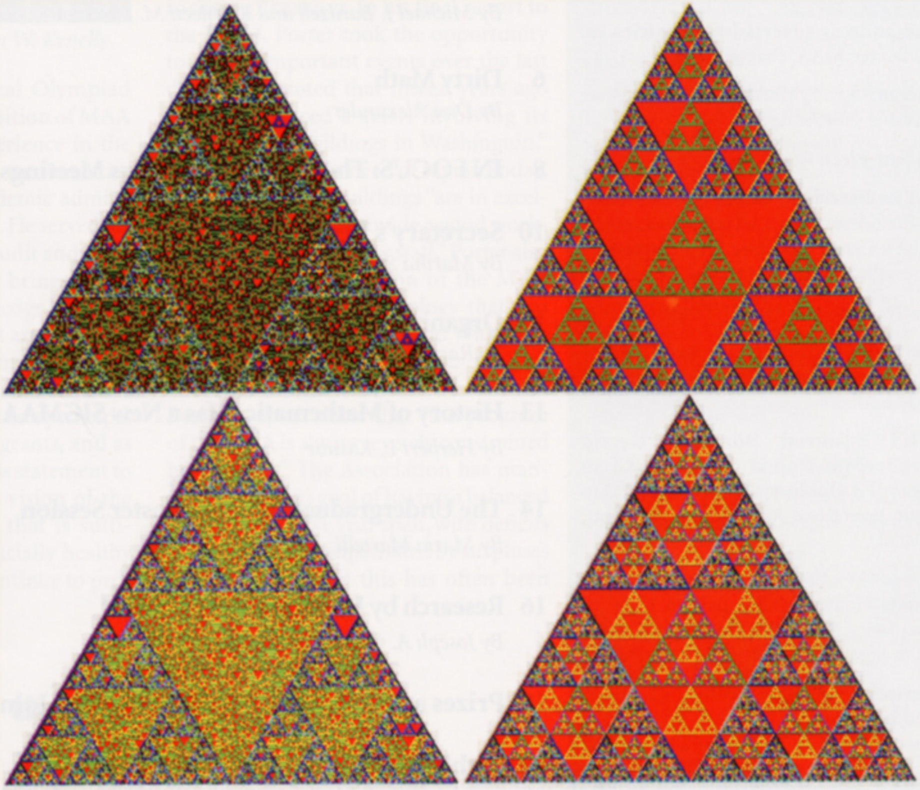


Does it matter if it's a p-group?

INSIDE:

The PascGalois Project: Visualizing Abstract Algebra

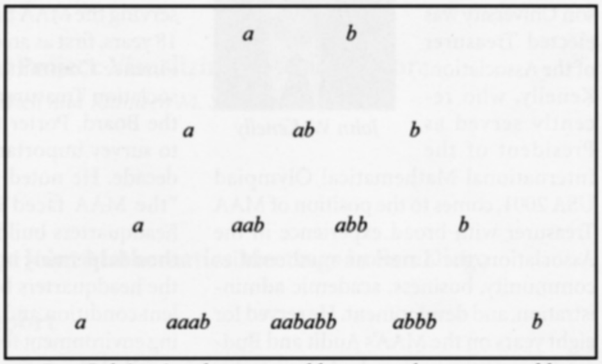
By Michael J. Bardzell and Kathleen M. Shannon

Abstract algebra has traditionally been one of the most difficult and least visual subjects in the undergraduate mathematics curriculum. Many mathematics majors find it their least favorite course. To help "repair" the image problem that abstract algebra has had, the newest generation of modern algebra textbooks include more emphasis on symmetry and applications such as coding theory, Cayley graphs, crystallography, boolean algebras, etc. (see [3] for example).

With the availability of computers in the classroom and computer algebra systems, there has also been an increased interest in computational techniques for polynomial rings and their applications to algebraic geometry at the undergraduate level (see [2]). The increased emphasis on symmetry, algebraic geometry, etc., is allowing some students to see more visual connections to abstract algebra. Currently we are engaged in an NSF funded project, *The PascGalois Project*, to introduce a new class of examples for undergraduate algebra that will also focus on visualization. But we are not just focusing on objects related to algebra. We are also interested in providing ways to visualize fundamental algebraic concepts as well — notions such as closure, subgroups, and even quotient groups.

The *PascGalois Project* has its origin in a simple exercise with Pascal's triangle. Take each entry in the triangle and replace it with its value mod *n*, where *n* is a positive integer larger than 1. By assigning each of the values 0,1,…,*n*-1 a distinct color, patterns reminiscent of fractals appear. Our interest in this construction lies in the fact that addition mod *n* is the group multiplication of the cyclic group Z_n and the patterns seen in the triangles are related to the structure of these groups. These patterns have been studied in the literature; there, the triangles are often treated as a type of 1-dimensional cellular automata (see [4] and [5]). Cellular automata consist of a discrete lattice of cells where each cell can take on values from some alphabet *A*. The cells are updated in discrete time steps according to some local rule — that is, the value of a given cell at time *t* is a function of the values of its neighboring cells at time $t - 1$. In the case of Pascal's triangle mod *n*, each successive row corresponds to the next time frame. The local rule in this case is simply "add the two entries above" for the current cell value. We can think of all the cells outside the triangle as being zero.

It is easy to generalize this construction using other groups. If *G* is a group with *a* and *b* elements of *G*, then a *PascGalois* triangle is formed by placing *a* down the left side of the triangle and *b* down the right. An entry in the interior of the triangle is determined by multiplying the two entries above it using the group multiplication. Of course, if *G* is non-abelian then one must specify a left or right multiplication. We denote this PascGalois triangle by (P_G, a, b). Like Pascal's triangle mod *n*, PascGalois triangles can have self-similar properties. Furthermore, many of these properties can be described using subgroups, quotients, and automorphisms of the group *G*.



PascGalois triangle generated by group elements a and b.

To see how these structures can be used to visualize algebraic concepts, let us consider the triangle (P_G, a, b) where $G = D_3$, the symmetry group of an equilateral triangle, *a* corresponds to a reflection and *b* to a 120 degree rotation. If we form the quotient group modulo the rotational subgroup of order three, we obtain a group isomorphic to Z_2. We can visualize this by coloring all the rotations in the triangle one color and all the reflections a second color. This helps the student "see" how one can identify all the elements in a coset into a single point in a quotient group. Thinking of a composite structure such as an equivalence class as a point in a new abstract group can be difficult for many students. This exercise helps to visually reinforce the concept.

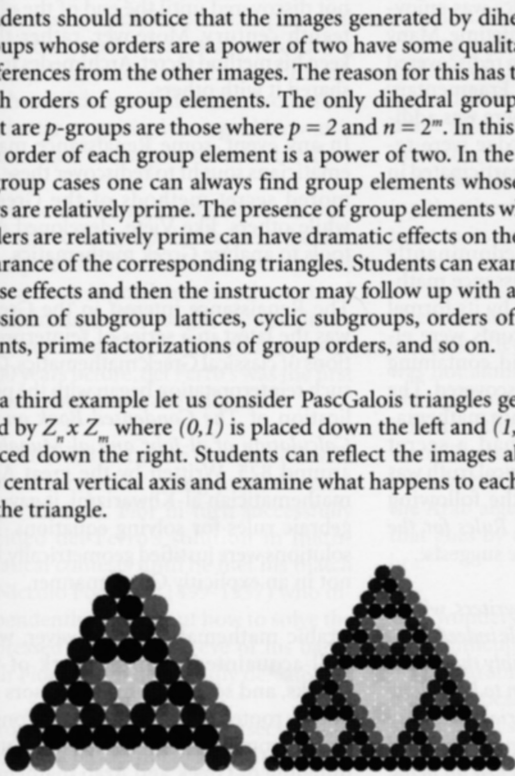
First 64 rows of the triangles corresponding to D_3 (left) and Z_2 (right). The Z_2 triangle has 1 down the left side and 0 down the right. The quotient identification modulo the subgroup of order 3 consisting of all rotations transforms the left triangle into the right triangle.

As a second example, consider the following images generated by dihedral groups, where we place a reflection down the left side of the triangle and the minimal nonzero rotation down the right:

Triangles generated by the dihedral groups D_4, D_6 and D_8, respectively.

Students should notice that the images generated by dihedral groups whose orders are a power of two have some qualitative differences from the other images. The reason for this has to do with orders of group elements. The only dihedral groups D_n that are p-groups are those where $p = 2$ and $n = 2^m$. In this case the order of each group element is a power of two. In the non 2-group cases one can always find group elements whose orders are relatively prime. The presence of group elements whose orders are relatively prime can have dramatic effects on the appearance of the corresponding triangles. Students can examine these effects and then the instructor may follow up with a discussion of subgroup lattices, cyclic subgroups, orders of elements, prime factorizations of group orders, and so on.

As a third example let us consider PascGalois triangles generated by $Z_n \times Z_m$ where $(0,1)$ is placed down the left and $(1,0)$ is placed down the right. Students can reflect the images about the central vertical axis and examine what happens to each cell in the triangle.



The first nine (left) and 16 (right) rows of triangles generated by $Z_2 \times Z_2$, $(0,1)$ is down the left side of the triangle and $(1,0)$ is down the right.

Upon experimentation one sees that this reflection induces the automorphism that maps (r,s) to (s,r) if and only if $n = m$. If n and m are not equal, then reflection does not even induce a set map. That is, two different locations of the same group element (color) can be reflected to 2 distinct group elements. For a more detailed description of these and other exercises see [1].

We can also consider 2-dimensional cellular automata generated by a group multiplication. 2-D automata are rectangular grids of cells that take on various state values that change discretely over time according to some local rule. The 2-D automata from this project are multi-state variations of Conway's Game of Life. The Game of Life is known to have interesting dynamical properties using two states (dead or alive) and using local rules to update the automata in the next time frame. Using groups as an alphabet and group multiplication for the various local rules, the long-term behavior of these systems can often be understood in terms of the subgroup lattice of the underlying group and dimensions of the grid for the finite case.

The primary goal of this project is to develop exercises that will help undergraduate mathematics majors, including prospective secondary school teachers, to develop intuition about and visualize many of the fundamental concepts in abstract algebra. We are in the process of producing visualization materials that can be used for class demonstrations, student computer exercises, group projects, or even as the starting point for an undergraduate research project.

The focus will be on creating PascGalois triangles and other cellular automata generated using group and ring multiplication rules to aid in giving students a visual understanding of key concepts in abstract algebra. A secondary goal is to provide projects for undergraduate research.

If you are interested in learning more, visit our web site at http://faculty.salisbury.edu/~kmshannon/pascal/. This site also contains numerous color images that reveal details that are much more difficult to see in the black and white images shown here.

We will, using support from NSF CCLI award # DUE-0087644, be offering a limited number of stipends to faculty who will participate in the evaluation of these materials in Fall 2002. If you would like to participate please send an e-mail to mjbardzell@salisbury.edu. ■

References:

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[4] Granville, Andrew; "Zaphod Beeblebrox's Brain and the Fifty-ninth row of Pascal's Triangle", *American Mathematical Monthly*, Volume 99, Number 4, April 1992; pp. 318-331.

[5] Wolfram, Steven; "Geometry of Binomial Coefficients"; *American Mathematical Monthly*, Volume 91, November 1984, pp. 566-57.