Unimodality of the binomial coefficients

Pascal's triangle is a nice way to organize the binomial coefficients and its structure has led mathematicians to discover many interesting properties and identities involving the binomial coefficients. For instance, if we focus on one particular row, say row n, we see that the binomial coefficients steadily increase to a maximum value and then decrease steadily (to the first value of that row). A sequence that has this property is said to be *unimodal*. If we were to plot the terms of a unimodal sequence we would see that they follow a bell-like curve.

Formally, we say that a sequence $a_0, a_1, a_2, \ldots, a_n$ is unimodal if there exist indices i, j such that

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_i = a_{i+1} = \cdots = a_{i+j} \geq a_{i+j+1} \geq \cdots \geq a_n.$$

Our goals in this lab are to prove that the sequence of binomial coefficients is unimodal and to establish bounds for the largest term in the sequence when n is even.

Exercises

1. Use the PascGalois JE software to see that Pascal's triangle is symmetric (make sure to pick a very large modulus, say 1000 or more, as the binomial coefficients grow very quickly). Show that $\binom{n}{k} = \binom{n}{n-k}$.

2. Use the PascGalois JE software to see that each row of Pascal's triangle has a maximum (and that it is achieved at least once). What is the parity of the rows that have the maximum value occurring only once? twice?

3. Show that if $k < (n-1)/2$, then $\binom{n}{k} < \binom{n}{k+1}$ and that if $k > (n-1)/2$, then $\binom{n}{k} > \binom{n}{k+1}$. What happens if $k = (n-1)/2$?

4. Show that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$. (*Hint*: Use the binomial theorem to expand $(x+1)^n$ and then let $x = 1$.) Use the above identity to show that $\binom{n}{n/2} < 2^n$.

5. Show that $\binom{n}{n/2} > \frac{2^n}{n+1}$. (*Hint*: What is $\frac{2^n}{n+1}$, the average of?)

6. Find the value of k for which $k\binom{546}{k}$ is largest.