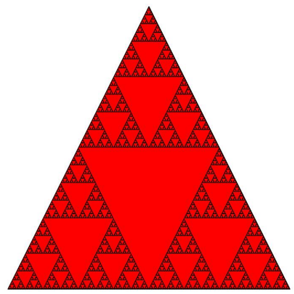
How many binomial coefficients are even?

It is no accident that the mod 2 Pascal's triangle bears a strong resemblance to Sierpinski's famous fractal called the *gasket*.



One can in fact think of the mod 2 triangle as a close relative to the gasket. The proof of this fact requires quite a bit of advanced number theory, so its proof will be omitted. (The interested reader may find a readable exposition of this fact in [2].) The gasket is the result of an iterative process where one starts with a filled-in equilateral triangle with side length s and removes the *middle* equilateral triangle formed by connecting the midpoints of each pair of sides. Proceed by removing the middle equilateral triangle in each of the three smaller remaining filled-in equilateral triangles (each with side length $\frac{1}{2}s$). Continuing this process indefinitely gives us the Sierpinski gasket.

An interesting question at this point is how much of the starting triangle did we remove? To simplify calculations, let's suppose that the starting triangle had area equal to 1. In the first step we removed a triangle that had area 1/4. In the second step we removed three triangles, each with area equal to 1/16. Thus the total removed area at this point is equal to $1/4 + 3/16$. Continuing this process we see that the removed area is equal to

$$1/4 + 3/16 + 9/64 + \cdots = 1.$$

That is, the remaining structure has zero area!

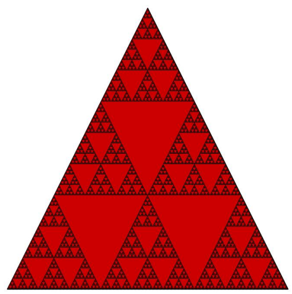
What does this calculation tell us about Pascal's triangle? To answer this question, first let P_r represent the first r rows of Pascal's triangle. Also let $n(P_r, 2)$ be the number of even binomial coefficients in P_r and let $t(P_r)$ represent the total number of binomial coefficients in P_r. Then this result tells us that

$$\lim_{r\to\infty} \frac{n(P_r, 2)}{t(P_r)} = 1.$$

That is, most of the binomial coefficients in Pascal's triangle are even, a surprising result. The exercises below ask you to see what else can be learned by looking at Pascal's triangle modulo an odd prime.

Exercises

1. Use the software PascGaloisJE to graph the mod 3 triangle. Make sure to color all of the nonzero residues the same (say black). The picture you obtain will look like

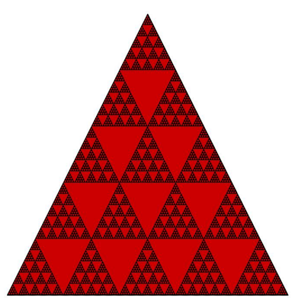


   (a) If the picture had infinite detail, like Sierpinski's gasket, how much area is *taken* away (i.e., colored red)?

   (b) What does this imply about the number of binomial coefficients that are divisible by 3?

   (c) How many rows of Pascal's triangle do we have to take to insure that the binomial coefficients that are divisible by 3 make up approximately 1/2 of the triangle?

2. Next use the software PascGaloisJE to graph the mod 5 triangle. Again, be sure to color all of the nonzero residues the same. The picture you obtain will look like



   (a) If the picture had infinite detail, like Sierpinski's gasket, how much area is *taken* away (i.e., colored red)?

(b) What does this imply about the number of binomial coefficients that are divisible by 5?

(c) How many rows of Pascal's triangle do we have to take to insure that the binomial coefficients that are divisible by 5 make up approximately 1/2 of the triangle?

3. Is what you found in exercises 1 and 2 true for larger primes? Can you prove it? What does this say about Pascal's triangle?

   A rigorous treatment of the results explored here can be found in [1].

References

[1] N. Fine, Binomial coefficients modulo a prime, *Monthly* **54** (1947), 589-92

[2] H-O. Peitgen, H. Jürgens, and D. Saupe , Chaos and Fractals: New Frontiers of Science, 2nd ed., *Springer-Verlag, New York, NY* (2004).