Exploiting Symmetry in Pascal's Triangle Modulo a Prime

The goal of this lab is to use patterns in Pascal's triangle modulo a prime to derive conjectures for some *binomial identities* that hold modulo a prime p. A binomial identity is a formula involving binomial coefficients. In your investigation, you will be interpreting a pattern in Pascal's triangle modulo p, and it will be useful to make additional identifications among the group elements. Your investigation will involve an application of the (action of the) dihedral symmetry group of the triangle.

Exercise 1 *Generate the first $p-1$ rows of Pascal's triangle modulo p for various primes p and classify the symmetry group of the resulting triangles. Repeat the above, but this time color every element a the same as its inverse $-a$.*

The groups you classified in exercise (1) *act by symmetries* on the set of entries, or cells, of the triangle — each of the symmetries of the triangle maps the set of cells back onto itself. The *orbit* of any cell has at most six elements — that is a given cell can be sent to up to as many as 6 distinct cells (including itself) under the group action. Cells in the same orbit will have the same color, and hence the binomial coefficients they represent in $\mathbb{Z}_p$ are equal (up to a sign in the case where you identified a with $-a$).

Recall the full symmetry group of the triangle, D, is generated by reflections in any two median lines. To understand the relationship between the binomial coefficients in a given orbit, it suffices to look at the orbits of cells under these two motions only. The image in Figure 1 shows one way to coordinatize the cells in the triangle (here we have taken $p = 13$).

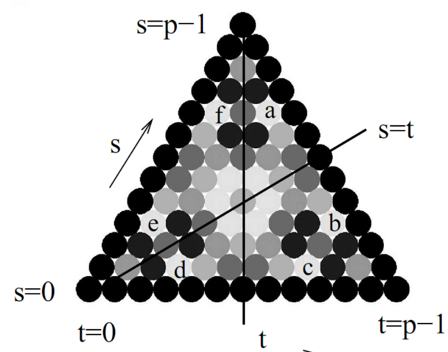

Figure 1: Each pair (s,t) $(s,t \geq 0$ and $s+t \leq p-1)$ corresponds to the cell $\binom{p-1-s}{t}$. The cell labeled with an "a" has coordinates $(s,t) = (8,3)$ and represents the residue class of $\binom{4}{3}$ (mod 13). The cells labeled "a" through "f" comprise an orbit and the binomial coefficients they represent are all congruent either to 4 or -4 modulo 13.

Exercise 2 *The vertical symmetry (which holds without the identification of the signs) gives a binomial identity. Find it. Does this symmetry depend on the sign identification?*

Exercise 3 *Prove the identity you conjectured in exercise (2).*

Exercise 4 *Referring to Figure 1, note that the symmetry across the line $s = t$ does depend on the signs. Look at Pascal's triangle modulo p and decide when the symmetry across the line $s = t$ depends on the sign identification. Write your result as a conjecture for another binomial identity.*

Exercise 5 *Use your conjecture from exercises (2) and (4) together with the fact that these two reflections generate the group D to conjecture three more (non-trivial) binomial identities.*

Exercise 6 *Prove the identities you conjectured in exercises (4) and (5).*