Pattern Recognition and Pascal's Triangle Modulo n

You may be familiar with Pascal's triangle and its construction using the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Using the fact that $\binom{n}{0} = \binom{n}{n} = 1$ for all $n \geq 0$, we see that Pascal's triangle has 1's down both sides of the triangle. Upon computing the rest of the entries we obtain the following:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

It is clear that any entry in the interior of the triangle is the sum of the two entries above it. This follows from the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. You may also have noticed that each row is symmetric, that is, a palindrome. This follows from the fact that $\binom{n}{k} = \binom{n}{n-k}$. The binomial theorem also states that the n^{th} row of Pascal's triangle gives the coefficients of the expansion for $(a+b)^n$:

$$
\begin{aligned}
(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n \\
&= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}
\end{aligned}
$$

In this activity we will study a modified version of this triangle called Pascal's triangle modulo n. Pick an integer $n \geq 2$. Then reduce each entry of the triangle modulo n. Note that any entry in the interior of this new triangle is the sum mod n of the two entries directly above it. Also note that some very interesting patterns emerge in this revised triangle. Of course these patterns depend upon which positive integer you picked initially. Since this construction can be time consuming to do by hand we use the PascGalois JE program to do it. This program will assign to each of the values $0, 1, \ldots, n-1$ a unique color.

For example, let $n = 3$, generate Pascal's triangle and then mod each entry by 3.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
\quad \overset{\text{mod}3}{\Longrightarrow} \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & 1 & & \\
 & 1 & & 1 & & 0 & & 1 & & 1 & \\
1 & & 2 & & 1 & & 1 & & 2 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Instructions:

Vocabulary:

As you begin this lab and go through it keep in mind the following concepts — congruence mod n and $\mathbb{Z}_n$, generator, prime number, composite number, identity.

Writing up your work:

As you go through this lab it will be helpful to take notes as you work through the material and receive new insights. Some of these questions are simple to answer, but others may require a little reflection. If you find yourself thinking "the answer to that is different for varying circumstances", then explain the answer for each circumstance.

All of your labs must be typewritten. Include all of the necessary setup information along with all supporting images. Make sure that your name is on your work; it would best to place your name and the lab assignment number in the header of each page (besides the fact that it looks really neat).

Exercises:

1. Using the PascGalois JE software, construct Pascal's triangle modulo 2. Recall from the software introduction lab that you need to select the group as Zn and input 2 for n, the default options of an infinite automata, pin alignment, 100 rows and a default element of 0 will be fine. Now select the Image tab and click the Refresh/Apply button.

   (a) What is the group under consideration?

   (b) How many elements are there in this group? How many in the triangle?

   (c) To better investigate patterns in the triangles, sometimes it is helpful to generate the triangles for a variable number of rows. Try doing this for Pascal's triangle modulo 2 and keep track of the patterns you observe and identify in writing what they are. You may find it helpful to use multiple windows.

   (d) Notice that with some choices of the number of rows the image appears to be "incomplete". That is, there are partial triangles at the bottom. How many rows does it take so that there are no partial triangles at the bottom? What is the first number of rows this happens on? What is the second number of rows this happens on? The third? and so on. Is there a pattern to the number of rows needed to make these complete triangles? If so, what is it?

   (e) What is the identity for this group? When does it show up in the triangle? (Keep in mind how Pascal's triangle is created when answering this).

2. Construct Pascal's triangle modulo n for $n = 3, 4, \ldots, 12$. Consider the questions posed in Exercise #1 for each value of n while exploring the triangles for these moduli. Also, experiment with these different triangles (perhaps using a different number of rows, coloring schemes and by using some of the techniques discussed in the introduction).

   (a) How does n being prime or composite affect the appearance of the corresponding triangle? Make sure to justify your response with images of the triangles.

   (b) Which values of n generate symmetric triangles? Explain. Test your hypothesis for larger values of n.

   (c) Are there certain values of n which produce triangles with more "discordant" patterns? Explain.

   (d) Make a hypothesis about how the prime factorization of n affects the appearance of the corresponding mod n triangles.

3. Colorings. Construct Pascal's triangle for $n = 3, \ldots, 12$. Using the color highlighting option discussed in the introductory lab to color all the nonidentity elements a single color. Recall that to do this you select the identity element from the element list to the right and then select Color $\succ$ Highlight

Elements. What happens to the appearance of the triangle? Does this confirm or enhance any insights you had earlier? Explain. If you had any new insights about the affects the prime factorizations of n, explain them here.