Building A Group with PascGalois Triangles

In this project you will build a group out of the PascGalois triangles. So far these triangles have been complicated structures with a plethora of internal detail. However, in this lab you will think of each triangle as a mere “point” or “element” in a more abstract structure. The addition of these triangles will then be visualized using the superimposer applet. This project should help you see that mathematical structures can often have many “layers”. That is, a structure can have intricate internal detail when viewed from one perspective, but then appear point-like or atomic when viewed from another. Although PascGalois triangles are complex when viewed internally, they will appear to be point-like in the group you are about to form from them. Understanding the different layers of abstraction for various structures is one of the key goals of mathematics. It can also be a very powerful tool. This way of thinking is essential for every mathematics major.

Let G be any group, define $P_{a,b}$ to be the PascGalois triangle that uses the group elements a and b as the seed. More specifically, the seed is a and b and the update rule is the standard Pascal's triangle rule where addition of triangle entries is replaced by the group operation of G. Now consider

$$T = \{P_{a,b} \mid a, b \in G\}$$

T can be thought of as the set of all possible PascGalois triangles that can be generated by the group G. In this lesson we will see that T can be given an algebraic structure of its own. If $P_{a,b}$ and $P_{c,d}$ are in T we can “multiply” $P_{a,b}$ and $P_{c,d}$ by defining $P_{a,b} * P_{c,d} = P_{ac,bd}$ where ac and bd are computed using the group operation of G.

Exercises:

1. Before proving some theorems in general we will look at a few examples. For our first example we will use $G = \mathbb{Z}_6$ and construct $P_{1,1}$. In the PascGalois JE program set the group to $\mathbb{Z}_6$. Recall that this is done by simply setting the group type to Zn under addition and setting the “n =” to 6. Now click the 1-D Automata tab. The options tab should currently be visible, we will leave most of the the default options as they are except that we will change the number of rows to 31. Now we will input the seed. Click on the Seed tab and note that the default has just a single 1 in the seed table. Set the number of columns of the seed table to 2 and then input a 1 into the blank cell of the table. At this point we are ready to graph the automta image. Click on the Image tab and then click on the Refresh/Apply button in the upper right corner of the applet. The division between the image and the color scheme is movable, so if the image is too small you can increase the size of the image window. Make sure that you reclick the Refresh/Apply button to resize the image. Now create $P_{1,3}$, $P_{2,3}$, $P_{2,4}$ and $P_{3,3}$. For $G = Z_6$, how many elements are in T?

   Now consider $G = Z_2$,

   (a) What are all of the combinations of possible seeds a and b?

   (b) For $G = Z_2$, how many elements are in T?

   (c) Construct each element of T. Are there any two elements that are the same? You may want to zoom in on the top of the triangle to help answer this question.

   (d) Construct a group operation table for the $*$ operation on T. This can be done in two ways. You could either create the table using the notation $P_{a,b}$ or you could use the images that are produced from the program. We would suggest using the images.

   (e) Is there an element that acts as an identity? If so, what is it? Also, if there is an identity, does each element have an inverse? If so, list the inverse of each element. Is there any relationship between the inverse of an element and the generators a and b of the triangles?

   Now consider $G = D_4$, recall that D_4 is the group of symmetries of the square.

   (a) Create $P_{R1,F3}$, $P_{F1,F3}$, $P_{R2,R3}$ and $P_{F3,R3}$.

(b) For $G = D_4$, how many elements are in T?

(c) What are $F1R2$ and $F3R3$? Construct $P_{F1R2,F3R3}$.

(d) What are $R2F1$ and $R3F3$? Construct $P_{R2F1,R3F3}$.

(e) Is T under the $*$ operation an abelian group?

2. Prove that under the operation $*$ defined above, T forms a group. If G is abelian, show that T is also. Recall that to answer this question completely you need to check that:

   (a) The operation $*$ is a binary operation on T.

   (b) The operation $*$ is associative.

   (c) There is an identity element for T under $*$.

   (d) Given any $P_{a,b}$ in T, there exists an inverse element $P_{c,d}$ in T.

   (e) And to verify that T is abelian whenever G is abelian you must further verify that $P_{a,b} * P_{c,d} = P_{c,d} * P_{a,b}$ for all $a, b, c, d \in G$.

3. If G is finite, what can you say about the order of T?

4. Prove that T? $G \times G$.

5. How does the internal structure of each triangle (for example, self-similarity, level of organization, etc.) affect your solutions for the above three exercises?

6. We have defined our operation on T in terms of the generators down the sides of the triangle. Now we will investigate what would happen if we defined a superposition operation $\otimes$ where $P_{a,b} \otimes P_{c,d}$ is defined to be the triangle obtained by multiplying corresponding entries of $P_{a,b}$ and $P_{c,d}$ using the group operation of G. That is, if $g \in G$ is in the i^{th} row and j^{th} column of $P_{a,b}$ and $h \in G$ is in the i^{th} row and j^{th} column of $P_{c,d}$ then $gh \in G$ is in the i^{th} row and j^{th} column of $P_{a,b} \otimes P_{c,d}$. We need to be careful with this definition. First of all, is $\otimes$ well-defined? In other words, is $P_{a,b} \otimes P_{c,d}$ necessarily an element of T? If not, then we do not have a binary operation. Even if $\otimes$ is a binary operation, is it the same as $*$? The purpose of the following exercises is to answer these questions.

   We will first look at an example before moving on to a general proof. First you will need the PascGalois JE Superimpose application. Since this superimpose operation is a rather specialized operation we have developed a separate application for the operation. When you start the program you will see that it uses the same type of tabbing system as the PascGalois JE program.

   We will now use the Superimpose application to draw $P_{1,2}$, $P_{2,4}$ and $P_{1,2} \otimes P_{2,4}$ using Z_5 under addition. Start the Superimpose application it should be on the Group tab when it starts. Select Zn under addition and set n to 5. We will leave the options at their defaults. Click on the Seeds tab and notice that we have two seed tables instead of just one. Set the number of columns to 2, note that both tables change simultaneously. In the first table, input the seed values of 1 and 2 and in the second seed table input the seed values of 2 and 4. Now click on the Images tab. Notice that there are three image windows and the color correspondence list on this tab. Click the Refresh/Apply button. At this point you should see $P_{1,2}$ in the first window, $P_{2,4}$ in the second window and $P_{1,2} \otimes P_{2,4}$ in the third. As with the PascGalois JE program the divider between the color correspondence list and the images is movable. Just make sure that you click the Refresh/Apply button after you resize the windows. The Superimpose application has many of the same features as the PascGalois JE program, for example, you have the same color grouping options and zooming options. The only difference here is that the changes will be applied to all three images. Furthermore, we have included a help system tab if you would like to investigate options of the program that we have not included in this lab. The main difference between this application and the PascGalois JE program is that we have not incorporated file or clipboard options. To paste images into a lab report we suggest using a screen dump, such as Alt+Print Screen in Windows or KSnapshot in Linux.

   Now, under our definition of $\otimes$ for these triangles, we see that $P_{1,2} \otimes P_{2,4} = P_{3,1}$. Next have the PascGalois JE program $P_{3,1}$. You should see that the $P_{3,1}$ triangle is the same as the triangle you obtained by superimposing the $P_{1,2}$ and $P_{2,4}$ triangles. Try this for other combinations as well. That is, superimpose other PascGalois triangles constructed over Z_5.

7. Prove that what occurred in the previous exercise happens in general for any abelian group G and $a, b, c, d \in G$. That is, show that by taking the n^{th} row of $P_{a,b}$ to the n^{th} row of $P_{c,d}$ produces the nth row of $P_{ac,bd}$. What can you say about $*$ and $\otimes$ in this case?

   Hint: For any abelian group G and $a, b \in G$, the n^{th} row of $P_{a,b}$ is

   $$a \quad a^{\binom{n}{1}}b^{\binom{n}{0}} \quad a^{\binom{n}{2}}b^{\binom{n}{1}} \quad a^{\binom{n}{3}}b^{\binom{n}{2}} \quad \ldots \quad a^{\binom{n}{n}}b^{\binom{n}{n-1}} \quad b$$

8. Now draw the D_4 triangles with generators $R1$, $F0$ and then $R2$, $F1$. Note that $R1$ is a counter-clockwise rotation by 90 degrees, $R2$ is a counter-clockwise rotation by 180 degrees, $F0$ is a flip over the horizontal and $F1$ is a flip over the line $y = x$. Use the Superimpose program to graph $P_{R1,F0}$, $P_{R2,F1}$ and $P_{R1,F0} \otimes P_{R2,F1}$. Use the PascGalois JE program to to graph $P_{R1R2,F0F1}$. Is the superimposed triangle $P_{R1,F0} \otimes P_{R2,F1}$ the same as $P_{R1R2,F0F1}$? If the two triangles are the same, explain why. If they are different, explain why superposition for D_4 triangles does not work like it does for triangles generated by Z_n. What can you say about $\otimes$ in this case?