Pattern Recognition Using Dihedral Groups

We are now ready to generalize the construction for Pascal's triangle mod n. Motivated by the fact that addition mod n is the group "multiplication" for the cyclic group Z_n; we now take any finite group G and let $a, b \in G$. A *PascGalois triangle* is formed by placing a down the left side of an equilateral triangle and b down the right. An entry in the interior of the triangle is determined by multiplying the two entries above it using the group multiplication. Of course, if G is nonabelian then one must specify a left or right multiplication. We denote this PascGalois triangle by (P_G, a, b) or as $P_{a,b}$ if the group G is understood. Note that Pascal's triangle mod n is $(P_{\mathbb{Z}}, 1, 1) = P_{1,1}$. The general construction of $P_{a,b}$ is the following:

$$
\begin{array}{ccccccccccc}
 & & & & a & & b & & & & \\
 & & & a & & ab & & b & & & \\
 & & a & & a^2b & & ab^2 & & b & & \\
 & a & & a^3b & & a^2bab^2 & & ab^3 & & b & \\
a & & a^4b & & a^3ba^2bab^2 & & a^2bab^2ab^3 & & ab^4 & & b \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

The word PascGalois comes from splicing Pascal with Galois. Galois (1811–1832) pioneered the study of groups while working with the theory of equations. Like Pascal's triangle mod n; PascGalois triangles can have many interesting patterns and self-similar properties. One of the goal's of this project, and the subsequent ones as well, is to understand these patterns in terms of group structure. Hopefully, studying the plethora of patterns within these triangles will provide you with a mechanism to visualize many of the fundamental concepts from abstract algebra. In this lab we will consider the dihedral groups D_n for $n \geq 3$ and introduce the concept of a p-group. Dihedral groups are the symmetry groups of regular polygons. D_3 is the symmetry group of an equilateral triangle, D_4 is the symmetry group of a square, D_5 is the symmetry group of a regular pentagon, and so on. If M is an n sided regular polygon, then M has n rotational symmetries and n reflectional symmetries. Hence the corresponding symmetry group has $2n$ elements. For example, let us consider the symmetry group of a square with corners labeled 1, 2, 3 and 4.

$$
\begin{array}{ccc}
1 & — & 2 \\
| & & | \\
4 & — & 3
\end{array}
$$

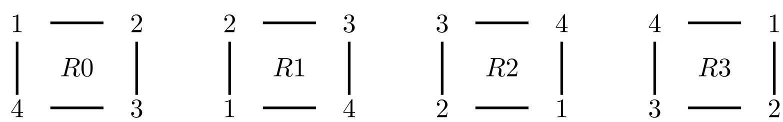
By turning the square counterclockwise, we see that there are 4 rotational symmetries: $0°$, $90°$, $180°$ and $270°$, shown below

$$
\begin{array}{ccc}
1 & — & 2 \\
| & & | \\
4 & — & 3
\end{array}
\qquad
\begin{array}{ccc}
2 & — & 3 \\
| & & | \\
1 & — & 4
\end{array}
\qquad
\begin{array}{ccc}
3 & — & 4 \\
| & & | \\
2 & — & 1
\end{array}
\qquad
\begin{array}{ccc}
4 & — & 1 \\
| & & | \\
3 & — & 2
\end{array}
$$

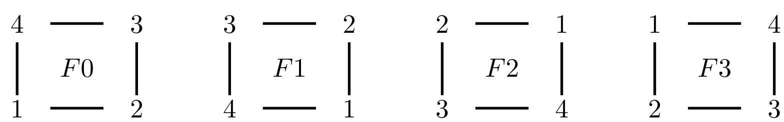
Of course $360°$, $450°$, ... are also symmetries, but they are equivalent to one of the four already listed. Note that we could also have done our rotations clockwise. We will use the convention that all the rotations will be counterclockwise since each clockwise rotation is equivalent to a counterclockwise rotation. Can you see why? We also see that there are four axes for reflectional symmetries. There is a vertical axis that bisects the square. Reflecting about this axis interchanges corners 1 and 2 and also 3 and 4. There is an analogous horizontal axis. Reflection about it interchanges 1 and 4 and also 2 and 3. Finally, there are two axes passing through the diagonals of the square. One passed through corners 1 and 3. Reflection about this axis leaves corners 1 and 3 fixed but interchanges 2 and 4. You should be able to describe the last axis at this point.

$$
\begin{array}{ccc}
2 & — & 1 \\
| & & | \\
3 & — & 4
\end{array}
\qquad
\begin{array}{ccc}
4 & — & 3 \\
| & & | \\
1 & — & 2
\end{array}
\qquad
\begin{array}{ccc}
1 & — & 4 \\
| & & | \\
2 & — & 3
\end{array}
\qquad
\begin{array}{ccc}
3 & — & 2 \\
| & & | \\
4 & — & 1
\end{array}
$$

The notation for the group elements of D_n vary from text to text. Usually the first non-trivial rotation is denoted with a character like ρ and hence all of the rotations can be written as $\rho^0 = e, \rho, \rho^2, \ldots, \rho^{n-1}$. The reflections are sometimes written with their own letter, such as, α, β, τ, ... and some texts will use a letter like τ to represent the reflection over the horizontal and then they write the remaining transformations as combinations of τ and ρ. When it comes to a computer program we need another notation. The PascGalois JE program uses an R for a rotation and F for a reflection (or flip). These letters are followed by a number that defines which rotation and which reflection it is. For the rotations of D_n the notation is $R0$, $R1$, $R2$, ..., $Rn-1$. $R0$ is the rotation by 0 degrees, that is, the identity. $R1$ is the rotation by $\frac{360}{n}$ degrees, $R2$ is the rotation by $2 \cdot \frac{360}{n}$ degrees and in general Ri is the rotation by $i \cdot \frac{360}{n}$. For D_4, $R1$ is the rotation by 90°, $R2$ is the rotation by 180°, and $R3$ is the rotation by 270°. Pictorially,



$F0$ is the reflection over the horizontal. $F1$ is the reflection over the line through the center of the polygon that makes an angle of $\frac{360}{2n}$ degrees with the horizontal. $F2$ is the reflection over the line through the center of the polygon that makes an angle of $2 \cdot \frac{360}{2n}$ degrees with the horizontal and in general Fi is the reflection over the line through the center of the polygon that makes an angle of $i \cdot \frac{360}{2n}$ degrees with the horizontal. For D_4, $F0$ is the reflection over the horizontal. $F1$ is the reflection over the line that is 45° from the horizontal. $F2$ is the reflection over the line that is 90° from the horizontal, that is, the vertical. $F3$ is the reflection over the line that is 135° from the horizontal. Pictorially,



When you are working with a polygon with an even number of vertices place half of the vertices above the horizontal and half below. When working with a polygon with an odd number of vertices place one vertex on the positive horizontal axis.

We say a group G is a p-group if $|G| = p^n$ for some prime p. An equivalent definition is that G is a p-group if, given any $a \in G$, $|a| = o(a) = p^m$ for some nonnegative integer m. Here p is a fixed prime. For example the cyclic groups Z_{17}, Z_4, Z_{27} and Z_{125} are all p-groups. However, Z_6 is not a p-group. Why? Before continuing to the problems below answer the following:

Question: D_n is a p-group if and only if ______________________.

Exercises:

For the first few problems we will consider the dihedral groups D_3 and D_4.

1. Using D_3, if we let σ be the rotation by 120° what is the notation for σ in the PascGalois JE program? If we let μ be the reflection over the horizontal what is the notation for μ in the PascGalois JE program?

2. PascGalois JE, draw the first 16 and 32 rows of $P_{\mu,\sigma}$. Do you see any patterns? How do these images compare with the Pascal's triangle mod n pictures?

3. Now draw the first 64, 128, 256, and 512 rows of the same triangle. As you add more rows, what happens to the corresponding image? Are certain group elements clumping together in the triangles. If so, how does this relate to group structure? closure? Note: Later when we study quotient groups, it will be much easier to understand this triangle.

4. Repeat the above exercises for the D_4 triangle. Compare and contrast the images you see with the D_3 images. Can you make a conjecture regarding the differences between these two images?

5. Draw the first 64 and 128 rows of the D_5, D_6, D_7 and D_8 triangles. Compare and contrast the qualitative properties of these images. Does your conjecture in the previous exercise still seem to be true?

6. Earlier you should have given a necessary and sufficient condition for D_n to be a p-group. Which of the dihedral groups D_n, $3 \leq n \leq 8$, are p-groups? Does this seem to affect the appearance of the corresponding triangles?

7. Recall that two integers r and s are relatively prime if $\gcd(r, s) = 1$: Which of the dihedral groups in the above exercise have at least one pair of nonidentity elements whose orders are relatively prime and which do not?

8. A subgroup triangle of a PascGalois triangle is a triangle that contains only elements from a subgroup of the original group. For each of D_3, D_4, D_5, D_6, D_7 and D_8 write out all of the possible subgroups of the group. For each of the six PascGalois triangles identify the subgroup triangles. Are there any subgroups not represented by a subgroup triangle? If so which ones?

9. Look at the subgroup triangles for these six PascGalois triangles. Is there any correlation between the existence of group elements of relatively prime order and the appearance of the subgroup triangles?