Subgroups and Generators of $\mathbb{Z}_n$

As you know from class a subgroup of a group is a subset of elements from the group that, under the same operation of the group, produces a group structure itself. The study of a groups subgroups can tell us a lot of information about the group itself, as we will see in the subsequent labs.

Since the integers modulo n are probability the most familiar to us this lab will deal exclusively with $\mathbb{Z}_n$. As you do this lab keep in mind the following concepts: generator, subgroup, identity, closure.

Exercises:

1. First we should review what the PascGalois triangles look like. Use the PascGalois JE program to graph the PascGalois triangles of $\mathbb{Z}_n$ for $n = 2, 3, 4, \ldots, 12$.

2. For each of the groups $\mathbb{Z}_n$ for $n = 2, 3, 4, \ldots, 12$, do the following.
   (a) List all of the subgroups of the group.
   (b) Which of these subgroups are cyclic and which are not?
   (c) For each subgroup find a generating set of elements, make sure that this generating set is minimal.
   (d) Construct a subgroup lattice for the group.

3. From the answers to #2 is there any correlation between the subgroup lattice for the group and the complexity of the PascGalois triangle you graphed in #1 for that group? Specifically,
   (a) When does it appear that the PascGalois triangle is made up of "overlapping" triangles?
   (b) What do these "overlapping" triangles correspond to?
   (c) What seems to determine when there are no "overlapping" triangles?
   (d) Are there any exceptions to these observations? If so, in what cases?

4. The PascGalois JE program has a way to bring out some of these "overlapping" triangles you investigated in #3. One feature of the PascGalois JE program we have not used very much yet has been the Seed. If you click the Seed tab you will see in the grid that there is the single number 1, which is of course the element at the top of the triangle. Why do you suppose that we have chosen this as the default seed? We will look at a specific example. In exercise #2 you found all of the subgroups of $\mathbb{Z}_6$. Take one of the non-trivial subgroups, what is its generator? Now in the Seed Table replace the 1 by this generating element. Click on the Image tab and regraph the image. Does this image look like any of the $\mathbb{Z}_n$ triangles we have graphed before? If so which one? Do the same for the other non-trivial subgroup. From your investigations here complete the following sentence for each non-trivial subgroup of $\mathbb{Z}_6$.

   The group ____ has a subgroup, ________________, generated by ____ which is isomorphic to ____.

   Repeat this exercise for $\mathbb{Z}_8$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{12}$.

5. In exercise #3 you looked at the relationship between the subgroup lattice and "overlapping" triangles. The PascGalois triangles also display subgroups in another way, at least when the complexity of the triangles is not too great. Look at the triangles you graphed in #1, notice that there are less complex triangles inside the group triangle, these are pointing down. For each of the PascGalois triangles of $\mathbb{Z}_n$ for $n = 2, 3, 4, \ldots, 12$ select one of these triangles that point down and list all of the elements that appear inside this inner triangle. Does this subset correspond to a subgroup? Taking this one step further, is there any case where one of these "pointing down" triangles does correspond to a non-trivial subgroup H and inside of this triangle there are yet smaller triangles that correspond to a non-trivial subgroup of H? If so, give an example.

   In general, prove that any subrow of elements from a subgroup $H < G$ must generate a triangular region beneath it containing only elements from H. Relate the length of the subrow to the number of rows of the triangular region.