Direct Products and Automorphisms

Consider Pascal's triangle mod n. We have already observed that it is symmetric. This follows from the fact that Pascal's triangle is symmetric.

Question: What property of binomial coefficients causes this symmetry?

Another way we can interpret this symmetry is that reflecting the triangle about its central vertical axis induces the identity map $i : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $i(m) = m$ for every $m \in \mathbb{Z}_n$. That is, this reflection takes every color in the triangle to itself. Note that the identity map gives a (trivial) automorphism of $\mathbb{Z}_n$. More generally, let G be any abelian group and $a, b \in G$. If we place a down the left side of the triangle and b down the right, then the group multiplication induces the following,

$$\begin{array}{ccccccccccccc}
 & & & & a & & & & b & & & & \\
 & & & a & & & ab & & & b & & & \\
 & & a & & a^2b & & & & ab^2 & & b & & \\
 & a & & a^3b & & & a^3b^3 & & & ab^3 & & b & \\
 & & & & & & \vdots & & & & & & \\
a & a^{\binom{n}{1}}b^{\binom{n}{0}} & & a^{\binom{n}{2}}b^{\binom{n}{1}} & & \dots & & a^{\binom{n}{n-1}}b^{\binom{n}{n-2}} & & a^{\binom{n}{n}}b^{\binom{n}{n-1}} & & b &
\end{array}$$

Let V denote the central vertical axis that bisects $P_{a,b}$. If we reflect $P_{a,b}$ about V then $a \leftrightarrow b$ and

$$\begin{array}{rcl}
a^{\binom{n}{1}}b^{\binom{n}{0}} & \leftrightarrow & a^{\binom{n}{n}}b^{\binom{n}{n-1}} \\
a^{\binom{n}{2}}b^{\binom{n}{1}} & \leftrightarrow & a^{\binom{n}{n-1}}b^{\binom{n}{n-2}} \\
 & \vdots & \\
a^{\binom{n}{j}}b^{\binom{n}{j-1}} & \leftrightarrow & a^{\binom{n}{n-j+1}}b^{\binom{n}{j}}
\end{array}$$

Using the fact that $\binom{n}{k} = \binom{n}{n-k}$, we have $a^{\binom{n}{n-j+1}} = a^{\binom{n}{j-1}}$. So in general $a^{\binom{n}{j}}b^{\binom{n}{j-1}} = a^{\binom{n}{j-1}}b^{\binom{n}{j}}$. In other words, reflection about V just interchanges the exponents of a and b.

In this lab we will investigate the group $\mathbb{Z}_n \times \mathbb{Z}_m$ and note what happens when $P_{(1,0),(0,1)}$ is reflected about its central axis. Note that $\mathbb{Z}_n \times \mathbb{Z}_m$ is abelian so we can use the results from the previous paragraph.

Exercises:

1. Consider $P_{(1,0),(0,1)}$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Does reflecting this triangle induce a function $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$? That is, can one color in two different locations of the triangle ever be sent to two different colors? (Recall that a function must have a unique output for a given input). If a function is not induced can you say why? If a function is induced, is there anything special about this map (think in terms of preserving algebraic structure)?

   In the PascGalois JE program one of the group types is Zn X Zm under addition. Select this and input n and m in the boxes below (they should both be set to 2 by default). Click on the 1-D Automaton tab and note that the default element has been changed to (0,0). If you are using an older version of PascGalois JE the default element may not have changed. In this case click the Use Group Identity button and then (0,0) should be loaded. Since you will be answering questions about where specific elements are sent by a map you will be looking at specific colors inside the triangle more than noting the structure of the triangle. To do this you probably do not want to graph a lot of rows since this will make picking out individual colors more difficult. Hence we should probably set the number of rows to something like 10 or 15. Now click on the Seed tab, change the number of columns to 2 and input the elements (1,0) and (0,1) into the table. Note that you must have a comma between the elements in the ordered pair. Now select the Image tab and graph the automaton.

2. Consider $P_{(1,0),(0,1)}$ in $\mathbb{Z}_2 \times \mathbb{Z}_3$. Does reflecting this triangle induce a function $\phi : \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3$? If a function is not induced can you say why? If a function is induced, is there anything special about this map?

3. Now let us generalize the previous exercises to $G = \mathbb{Z}_n \times \mathbb{Z}_m$, where $n, m \geq 2$. Reflect $P_{(1,0),(0,1)}$ about its central axis for various values of m and n. Try several combinations of n and m until you understand how these values affect the triangle. Give a conjecture as to when this reflection induces a function $\phi : \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3$? Note your answer will depend on the choices for n and m. When a set map is obtained, what special properties do you think it has?

4. (Optional) Give a proof for your conjecture(s) in the previous exercise. You may want to use the result discussed above that $a^{\binom{n}{j}} b^{\binom{n}{j-1}} = a^{\binom{n}{j-1}} b^{\binom{n}{j}}$ for any a, b from an abelian group G.