Quotient Groups of $\mathbb{D}_n$ and $\mathbb{S}_n$

This lab is a continuation of the The Quotient Groups of $\mathbb{Z}_n$ lab. We would suggest completing that lab before atempting this one. The Quotient Groups of $\mathbb{Z}_n$ lab give us a good warm-up to the concept of a quotient group but there is one detail that we could not examine. Since we were working with $\mathbb{Z}_{10}$ and $\mathbb{Z}_{12}$ all of the subgroups were normal. What about groups that are not abelian?

The goal of this lab is to investigate how PascGalois triangles can be used to determine quotient groups of non-abelian groups and how they can be used to determine if a subgroup is normal or not. We will start with symmetric groups.

Exercises:

Part 1: $\mathbb{S}_n$

1. Graph $P_{(1\ 2),(1\ 2\ 3)}$ under $\mathbb{S}_3$.
2. Do the coset coloring for (1 2 3), and then click the Refresh/Apply button.
3. How many cosets are there and how many elements are in each coset?
4. How many elements are in $\mathbb{S}_3/\langle(1\ 2\ 3)\rangle$?
5. If $\mathbb{S}_3/\langle(1\ 2\ 3)\rangle$ is a group structure what must it be isomorphic to?
6. From the graph of $P_{(1\ 2),(1\ 2\ 3)}$ does the above isomorphism appear to be true? The answer to this relies, of course, on whether or not the image of the coset structure looks like the image of the group you specified in the last exercise.
7. Using the information in the last exercise, is $\langle(1\ 2\ 3)\rangle$ a normal subgroup of $\mathbb{S}_3$? Why or why not?
8. The last exercise had you determine normality by determining if the quotient structure had the same structure as a known group. As you know from class a normal subgroup can also be determined by examining whether or not the left cosets and the right cosets are equivalent. That is, $aH = Ha$ for all $a \in G$. The PascGalois JE program has two ways to help you determine normality this way as well. The first is watch the color correspondence as you change from left to right coset colorings (or right to left for that matter) and the second is to use the Calculator tab.

   For the first method, do a left coset coloring of the cosets of $\langle(1\ 2\ 3)\rangle$. Now select the right coset coloring and watch the color correspondence window. If the color scheme does not change then you have a normal subgroup but if you see the color scheme change then there is the possibility that the subgroup is not normal. To be sure that you have a non-normal subgroup you simply need to find a coset that changes. So look at the colors that change as you go from left to right cosets, if they are different then you know you have non-normality.

   For the second method, click on the Calculator tab and input the subgroup generators into the subgroup section and click on the generate subgroup button "$\langle a\rangle$" to generate the subgroup. Now pick an element not in the subgroup and input it into the representative line. Now click on the generate left coset button and then on the generate right coset button if the cosets differ then you have non-normality. Unfortunately, to verify normality this way you must check all possible cosets, which can at times be tedious but at least it is faster than doing it by hand.

   Using these two methods, verify your conclusion from the last exercise.
9. Repeat the above exercises with
   (a) the subgroup $\langle(1\ 2)\rangle$ of $\mathbb{S}_3$.
   (b) the subgroup $\langle(1\ 2\ 3\ 4)\rangle$ of $\mathbb{S}_4$.

(c) the subgroup $\langle(1\ 2\ 3)\rangle$ of $\mathbb{S}_4$.

(d) the subgroup $\mathbb{A}_4$ of $\mathbb{S}_4$. Recall that $\mathbb{A}_n$ is generated by all of the three cycles of $\mathbb{S}_n$ so to do your coset coloring simply select all of the three cycles in the color correspondence and then group the cosets. You may skip the Calculator method for this one.

Part 2: $\mathbb{D}_n$

This next sequence of exercises is to show that we may want to be a little careful before stating conclusions.

(a) Graph $P_{R1,F0}$ under $\mathbb{D}_4$ use 100 to 200 rows.

(b) Using the methods discussed in the last sequence of exercises determine if $\langle R1\rangle$ is a normal subgroup of $\mathbb{D}_4$ and if so determine what $\mathbb{D}_4/\langle R1\rangle$ is isomorphic to.

(c) Reset the color scheme of $P_{R1,F0}$.

(d) Use the left coset colorings on the subgroup $\langle F3\rangle$.

(e) How many elements are in $\mathbb{D}_4/\langle F3\rangle$?

(f) Using your answer to the last question, if $\mathbb{D}_4/\langle F3\rangle$ is a group what are the possible groups that it must be isomorphic to?

(g) Use other windows to graph all of the possibilities you came up with in the previous exercise. Does the structure of $\mathbb{D}_4/\langle F3\rangle$ look like any of the other graphs? If not, display where there is an inconsistency. Be careful, you may wish to do some zooming on the triangles. Use the color correspondence method or the calculator method to verify your conclusions.

Keep in mind that most methods in mathematics will give clear and easy to interpret information for some situations and not so easy to interpret information in others. Take for example the simple computer generated graph of a function. If you use your graphing calculator or even a computer algebra system like Maple or Mathematica to graph the two functions $f(x) = x - 2$ and $g(x) = \frac{x^2-5x+6}{x-3}$ you could easily come to the conclusion that these functions are identical.