Subgroups and Generators of $\mathbb{D}_n$ and $\mathbb{S}_n$

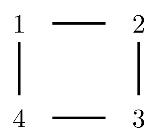
This lab is an extension of the Subgroups and Generators of $\mathbb{Z}_n$ lab. We would suggest doing that lab before attempting this one.

As you know from class a subgroup of a group is a subset of elements from the group that, under the same operation of the group, produces a group structure itself. The study of a groups subgroups can tell us a lot of information about the group itself, as we will see in the subsequent labs. As you do this lab keep in mind the following concepts: generator, subgroup, identity, closure.

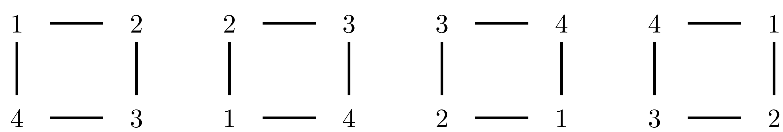
Part 1: $\mathbb{D}_n$

We will first look a what a Dihedral group is and discuss the notation that the PascGalois JE program uses for these groups. If you are already familiar with these group structures and know the notation used by the PascGalois JE program you may skip to the exercises.

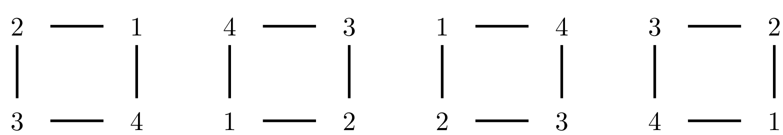
Dihedral groups are the symmetry groups of regular polygons. D_3 is the symmetry group of an equilateral triangle, D_4 is the symmetry group of a square, D_5 is the symmetry group of a regular pentagon, and so on. If M is an n sided regular polygon, then M has n rotational symmetries and n reflectional symmetries. Hence the corresponding symmetry group has $2n$ elements. For example, let us consider the symmetry group of a square with corners labeled 1, 2, 3 and 4.



By turning the square counterclockwise, we see that there are 4 rotational symmetries: 0°, 90°, 180° and 270°, shown below



Of course 360°, 450°, ... are also symmetries, but they are equivalent to one of the four already listed. Note that we could also have done our rotations clockwise. We will use the convention that all the rotations will be counterclockwise since each clockwise rotation is equivalent to a counterclockwise rotation. Can you see why? We also see that there are four axes for reflectional symmetries. There is a vertical axis that bisects the square. Reflecting about this axis interchanges corners 1 and 2 and also 3 and 4. There is an analogous horizontal axis. Reflection about it interchanges 1 and 4 and also 2 and 3. Finally, there are two axes passing through the diagonals of the square. One passed through corners 1 and 3. Reflection about this axis leaves corners 1 and 3 fixed but interchanges 2 and 4. You should be able to describe the last axis at this point.



The notation for the group elements of D_n vary from text to text. Usually the first non-trivial rotation is denoted with a character like ρ and hence all of the rotations can be written as $\rho^0 = e$, ρ, ρ^2, ..., ρ^{n-1}. The reflections are sometimes written with their own letter, such as, α, β, τ, ... and some texts will use a letter like τ to represent the reflection over the horizontal and then they write the remaining transformations as combinations of τ and ρ. When it comes to a computer program we need another notation. The PascGalois JE program uses an R for a rotation and F for a reflection (or flip). These letters are followed by a number that defines which rotation and which reflection it is. For the rotations of D_n the notation is $R0$, $R1$, $R2$, ..., $Rn-1$. $R0$ is the rotation by 0 degrees, that is, the identity. $R1$ is the rotation by $\frac{360}{n}$ degrees, $R2$ is the rotation by $2 \cdot \frac{360}{n}$ degrees and in general Ri is the rotation by $i \cdot \frac{360}{n}$. For D_4, $R1$ is the rotation by $90°$, $R2$ is the rotation by $180°$, and $R3$ is the rotation by $270°$. Pictorially,

$$\begin{array}{ccc} 1 & \text{---} & 2 \\ | & R0 & | \\ 4 & \text{---} & 3 \end{array} \qquad \begin{array}{ccc} 2 & \text{---} & 3 \\ | & R1 & | \\ 1 & \text{---} & 4 \end{array} \qquad \begin{array}{ccc} 3 & \text{---} & 4 \\ | & R2 & | \\ 2 & \text{---} & 1 \end{array} \qquad \begin{array}{ccc} 4 & \text{---} & 1 \\ | & R3 & | \\ 3 & \text{---} & 2 \end{array}$$

$F0$ is the reflection over the horizontal. $F1$ is the reflection over the line through the center of the polygon that makes an angle of $\frac{360}{2n}$ degrees with the horizontal. $F2$ is the reflection over the line through the center of the polygon that makes an angle of $2 \cdot \frac{360}{2n}$ degrees with the horizontal and in general Fi is the reflection over the line through the center of the polygon that makes an angle of $i \cdot \frac{360}{2n}$ degrees with the horizontal. For D_4, $F0$ is the reflection over the horizontal. $F1$ is the reflection over the line that is $45°$ from the horizontal. $F2$ is the reflection over the line that is $90°$ from the horizontal, that is, the vertical. $F3$ is the reflection over the line that is $135°$ from the horizontal. Pictorially,

$$\begin{array}{ccc} 4 & \text{---} & 3 \\ | & F0 & | \\ 1 & \text{---} & 2 \end{array} \qquad \begin{array}{ccc} 3 & \text{---} & 2 \\ | & F1 & | \\ 4 & \text{---} & 1 \end{array} \qquad \begin{array}{ccc} 2 & \text{---} & 1 \\ | & F2 & | \\ 3 & \text{---} & 4 \end{array} \qquad \begin{array}{ccc} 1 & \text{---} & 4 \\ | & F3 & | \\ 2 & \text{---} & 3 \end{array}$$

When you are working with a polygon with an even number of vertices place half of the vertices above the horizontal and half below. When working with a polygon with an odd number of vertices place one vertex on the positive horizontal axis.

1. We usually only work with $\mathbb{D}_n$ for $n \geq 3$. How many elements are in $\mathbb{D}_n$ for $n \geq 3$? Since $\mathbb{D}_n$ represents the symmetries of a regular n-gon what would the geometric object be for $\mathbb{D}_2$? Hence how many elements are in $\mathbb{D}_2$? So what is $\mathbb{D}_2$ isomorphic to? Answer the same questions for $\mathbb{D}_1$. Now that this is settled let's move onto more interesting investigations. So from here on out we will assume that $n \geq 3$.

2. Is $\mathbb{D}_n$ abelian for any $n \geq 3$? If so prove it and if not give a counter example.

3. Is $\mathbb{D}_n$ cyclic for any $n \geq 3$? Why or why not?

4. What elements generate $\mathbb{D}_n$? Is there a minimal set of generators? If so, what are they? Can you find more than one minimal set?

   In this exercise you may find the Calculator tab of some use. Doing calculations in $\mathbb{D}_n$ can be a bit tedious especially if you are drawing an n-gon for each operation. Each child window has a Calculator tab that will do quick calculations inside the group you have selected. This tab has three main sections, the Group Operation section, Subgroup section and Coset section. We will deal with the Coset section in another lab. In the Group Operation section you simply type in the group elements, using the PascGalois JE program's notation, into the two boxes and select the "*" button to do the operation. There is also a "^" key that allows you to take an element to a power, in this case the entry in the right hand box must be an integer. There is also an interchange key that will interchange the two elements so you can quickly check if the operation is commutative. The Subgroup section is particularly useful for this exercise, simply type in the elements separated by commas and click the generate subgroup button "$\langle a \rangle$". The list below will contain all of the elements in the subgroup generated by the elements you supplied.

5. Graph the PascGalois triangles of $\mathbb{D}_n$ for $n = 3, 4, \ldots, 8$. Recall from the introductory lab that you need to select the Group tab, select Dn in the drop-down selection box at the top and then input the value of n. Next select 1-D Automaton. If you are using a recent version of the PascGalois JE program you will notice that the Default Element has been changed from 0 to $R0$, which is the PascGalois JE notation for a rotation by $0°$, in other words the identity. If you are using an older version of the program this may still be 0. In which case click on the Use Group Identity button and then $R0$ should be loaded in. Next go to the Seed tab. Since 1 is not an element of $\mathbb{D}_n$ we will need to change the seed. In exercise #4 you should have found that $R1$ and $F0$ always work as a set of generators, there will of course be others. Change the number of columns in the seed table to two and then input these

two generators. At this point you can go to the Image tab and graph the image. Make sure that all of the elements of $\mathbb{D}_n$ show up in the color correspondence to the right. We will usually use these two generators for $\mathbb{D}_n$ but note what happens if we use a different set of generators. Take some of the generating sets you found in exercise #4 and use them as seeds. Are there any generating sets that do not produce the entire group when you create the PascGalois triangle? If so, why do you think this happened.

6. Can you see any "embedded" triangles in these examples? If so, list any "embedded" triangles you see. Do this for each value of n. Use these observations to complete the following sentence for each of the "embedded" triangles.

   The group ____ has a subgroup which is isomorphic to ____.

7. For each of the PascGalois triangles of $\mathbb{D}_n$ for $n = 3, 4, \ldots, 8$, do you see any of the "pointing down" triangles? Is there a relationship between seeing these and the value of n? If so, what are your observations? Is there any case where one of these "pointing down" triangles does correspond to a non-trivial subgroup H and inside of this triangle there are yet smaller triangles that correspond to a non-trivial subgroup of H? If so, give an example.

8. For each of the groups $\mathbb{D}_n$ for $n = 3, 4, \ldots, 8$, do the following. You may find the Calculator tab of some use.

   (a) List all of the subgroups of the group.

   (b) Which of these subgroups are cyclic and which are not?

   (c) For each subgroup find a generating set of elements, make sure that this generating set is minimal.

   (d) For each subgroup use the generating set to construct a PascGalois triangle. Does the image resemble any PascGalois triangle you have seen before? If so, complete the following sentence.

   The group ____ has a subgroup which is isomorphic to ____.

   (e) Construct a subgroup lattice for the group.

9. Is there any correlation between n and the complexity of the PascGalois triangle you graphed in #5 for that group? If so, what are your observations?

Part 2: $\mathbb{S}_n$

We will first look a what a Symmetric group is and discuss the notation that the PascGalois JE program uses for these groups. If you are already familiar with these group structures and know the notation used by the PascGalois JE program you may skip to the exercises.

S_n is the set of all permutations on n letters, usually called the Symmetric group. For the letters we usually use the numbers $\{1, 2, 3, \ldots, n\}$. One notation for a permutation is cycle notation which displays where each element in the permutation goes. For example, if we have the notation (1, 3, 4, 2) we know that the permutation sends 1 to 3, 3 to 4, 4 to 2 and 2 to 1. Any element not displayed in the cycle notation remains fixed by the permutation. Similarly, the notation (1 6 3)(2 5 4) sends 1 to 6, 6, to 3, 3 to 1, 2 to 5, 5 to 4 and 4 to 2. In this notation we can represent S_3 as { (1), (1 2), (1 3), (2 3), (1 2 3), (1 3 2) }. It can be shown that any permutation can be written as a product of disjoint cycles, so we can use this notation to represent any element of S_n.

The operation on this group is permutation, or function, composition done right to left. For example, in the calculation (1 2 4 3)(3 5 2 4) we start with the first number in the set "1". The right permutation does not have a 1 in it so it fixes 1, so we move on to the left permutation where 1 gets sent to 2. So to start our answer we have 1 going to 2, i.e. (1 2. Now where does 2 go? The right permutation sends 2 to 4 and the left permutation sends 4 to 3, so in all 2 goes to 3 and we have (1 2 3. Now we do the same for 3. The right permutation sends 3 to 5 and the left permutation fixes 5 so in all 3 goes to 5, giving (1 2 3 5. Next 5 goes to 2 and then 2 goes to 4 so in all 5 goes to 4, giving (1 2 3 5 4. Finally, 4 goes to 3 and then 3 goes to 1 so

4 goes to 1, which is the first letter in our list so we close the parentheses for a final answer of (1 2 3 5 4). So (1 2 4 3)(3 5 2 4) = (1 2 3 5 4). Some other examples, (1 4 2)(3 2 4) = (1 4 3), (1 2 4 3)(3 2 5) = (1 2 5)(3 4), and (1 2 4 3)(6 7 2)(3 2 1)(3 2 5) = (2 5 6 7 4 3).

The notation used by the PascGalois JE program in this case is just like the notation we would write by hand. Simply put all of the cycles in parentheses and make sure that there is at least one space between the "letters" in each cycle. It may be tempting to separate the letters by commas, do not do that, the PascGalois JE program will see this as a Cartesian product and give you a syntax error.

1. We usually only work with $\mathbb{S}_n$ for $n \geq 3$. How many elements are in $\mathbb{S}_n$ for $n \geq 3$? Since $\mathbb{S}_n$ represents the permutations on n letters what would $\mathbb{S}_2$ look like if we wrote all of the elements in cycle notation? So what is $\mathbb{S}_2$ isomorphic to? Answer the same questions for $\mathbb{S}_1$. Now that this is settled let's move onto more interesting investigations. So from here on out we will assume that $n \geq 3$.

2. Is $\mathbb{S}_n$ abelian for any $n \geq 3$? If so prove it and if not give a counter example.

3. Is $\mathbb{S}_n$ cyclic for any $n \geq 3$? Why or why not?

4. What elements generate $\mathbb{S}_n$? Is there a minimal set of generators? If so, what are they? Can you find more than one minimal set? In this exercise you may find the Calculator tab of some use.

5. Graph the PascGalois triangles of $\mathbb{S}_n$ for $n = 3, 4, 5$. Recall from the introductory lab that you need to select the Group tab, select Sn in the drop-down selection box at the top and then input the value of n. Next select 1-D Automaton. If you are using a recent version of the PascGalois JE program you will notice that the Default Element has been changed from 0 to (1), which is the PascGalois JE notation for the identity. If you are using an older version of the program this may still be 0. In which case click on the Use Group Identity button and then (1) should be loaded in. Next go to the Seed tab. Since 1 is not an element of $\mathbb{S}_n$ we will need to change the seed. In exercise #4 you should have found that (1 2) and (1 2 3) generate $\mathbb{S}_3$, there will of course be others. Change the number of columns in the seed table to two and then input these two generators. In general, the elements (1 2) and $(1\ 2\ \ldots n)$ generate $\mathbb{S}_n$. At this point you can go to the Image tab and graph the image. Do all of the elements of $\mathbb{S}_3$ show up in the triangle? What about with $\mathbb{S}_4$ and $\mathbb{S}_5$. If not, why do you think this happened? Can you find a "better" set of generators?

6. Can you see any "embedded" triangles in these examples? If so, list any "embedded" triangles you see. Do this for each value of n. Use these observations to complete the following sentence for each of the "embedded" triangles.

   The group ____ has a subgroup which is isomorphic to ____.

7. For each of the PascGalois triangles of $\mathbb{S}_n$ for $n = 3, 4, 5$, do you see any of the "pointing down" triangles? Is there a relationship between seeing these and the value of n? If so, what are your observations? Is there any case where one of these "pointing down" triangles does correspond to a non-trivial subgroup H and inside of this triangle there are yet smaller triangles that correspond to a non-trivial subgroup of H? If so, give an example.

8. Clearly, $\mathbb{S}_n$ is a far more complicated structure than $\mathbb{Z}_n$ or $\mathbb{D}_n$. The PascGalois triangles look far more chaotic then those produced by either $\mathbb{Z}_n$ or $\mathbb{D}_n$. This exercise will try to point out why.

   (a) List all of the subgroups of $\mathbb{S}_3$ and construct a subgroup lattice. How many subgroups does $\mathbb{S}_3$ have? What are the orders of each of these subgroups?

   (b) How many groups of order 2 are there? Does $\mathbb{S}_3$ have a subgroup that is isomorphic to any group of order 2? If so, which one(s)?

   (c) How many groups of order 3 are there? Does $\mathbb{S}_3$ have a subgroup that is isomorphic to any group of order 3? If so, which one(s)?

   (d) Now we will consider $\mathbb{S}_4$. Does $\mathbb{S}_4$ have a subgroup isomorphic to any group of order 2? Order 3?

(e) How many groups of order 4 are there? Does $\mathbb{S}_4$ have a subgroup that is isomorphic to any group of order 4? If so, which one(s)?

(f) Think about the generators to $\mathbb{S}_n$, in particular the standard generators of $\mathbb{S}_n$. Does $\mathbb{S}_4$ have a subgroup isomorphic to $\mathbb{S}_3$. If so, what elements of $\mathbb{S}_4$ generate this subgroup?

(g) Does $\mathbb{S}_5$ have a subgroup isomorphic to $\mathbb{S}_3$. If so, what elements of $\mathbb{S}_5$ generate this subgroup? Does $\mathbb{S}_5$ have a subgroup isomorphic to $\mathbb{S}_4$. If so, what elements of $\mathbb{S}_5$ generate this subgroup?

(h) Make a general conjecture about when $\mathbb{S}_n$ has a subgroup isomorphic to $\mathbb{S}_m$ and prove your conjecture.

(i) Does $\mathbb{S}_5$ have a subgroup isomorphic to $\mathbb{Z}_2$. If so, what elements of $\mathbb{S}_5$ generate this subgroup? What about subgroups isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_5$?

(j) Make a general conjecture about when $\mathbb{S}_n$ has a subgroup isomorphic to $\mathbb{Z}_m$ and prove your conjecture.

(k) If we consider $\mathbb{D}_n$ again, there is a nifty way that we can represent elements of $\mathbb{D}_n$ as permutations. Consider the two diagrams we used for $\mathbb{D}_4$.

$$\begin{array}{ccc} 1 & — & 2 \\ | & R0 & | \\ 4 & — & 3 \end{array} \qquad \begin{array}{ccc} 2 & — & 3 \\ | & R1 & | \\ 1 & — & 4 \end{array} \qquad \begin{array}{ccc} 3 & — & 4 \\ | & R2 & | \\ 2 & — & 1 \end{array} \qquad \begin{array}{ccc} 4 & — & 1 \\ | & R3 & | \\ 3 & — & 2 \end{array}$$

$$\begin{array}{ccc} 4 & — & 3 \\ | & F0 & | \\ 1 & — & 2 \end{array} \qquad \begin{array}{ccc} 3 & — & 2 \\ | & F1 & | \\ 4 & — & 1 \end{array} \qquad \begin{array}{ccc} 2 & — & 1 \\ | & F2 & | \\ 3 & — & 4 \end{array} \qquad \begin{array}{ccc} 1 & — & 4 \\ | & F3 & | \\ 2 & — & 3 \end{array}$$

We can create a permutation out of each of these elements by considering where each of the vertices go when we apply the transformation. For example, consider the original vertex numbering and the new numbering after applying $R1$.

$$\begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array} \qquad \begin{array}{ccc} 2 & — & 3 \\ | & R1 & | \\ 1 & — & 4 \end{array}$$

Vertex 1 goes to 2, 2 to 3, 3 to 4 and 4 to 1. This is the same as the permutation $(1\ 2\ 3\ 4)$. Similarly, for $F0$ we get 1 to 4, 4 to 1, 2 to 3 and 3 to 2, giving the permutation $(1\ 4)(2\ 3)$. Translate the other 6 transformations into permutations. So does $\mathbb{S}_4$ have a subgroup isomorphic to $\mathbb{D}_4$? If so, what are the generators of this subgroup?

(l) Do you think that $\mathbb{S}_3$ will have a subgroup isomorphic to $\mathbb{D}_3$? Can you find a set of generators for a subgroup of $\mathbb{S}_3$ that will be isomorphic to $\mathbb{D}_3$? How many elements are in $\mathbb{S}_3$? How many elements are in $\mathbb{D}_3$? What is all of this implying about $\mathbb{S}_3$ and $\mathbb{D}_3$? Create the PascGalois triangles (using the standard generators for each group), what observations can you make?

(m) Do you think that $\mathbb{S}_5$ will have a subgroup isomorphic to $\mathbb{D}_5$? Do you think that $\mathbb{S}_6$ will have a subgroup isomorphic to $\mathbb{D}_6$? Why or why not?

(n) We will not prove this here but the above questions are leading to what conjecture?