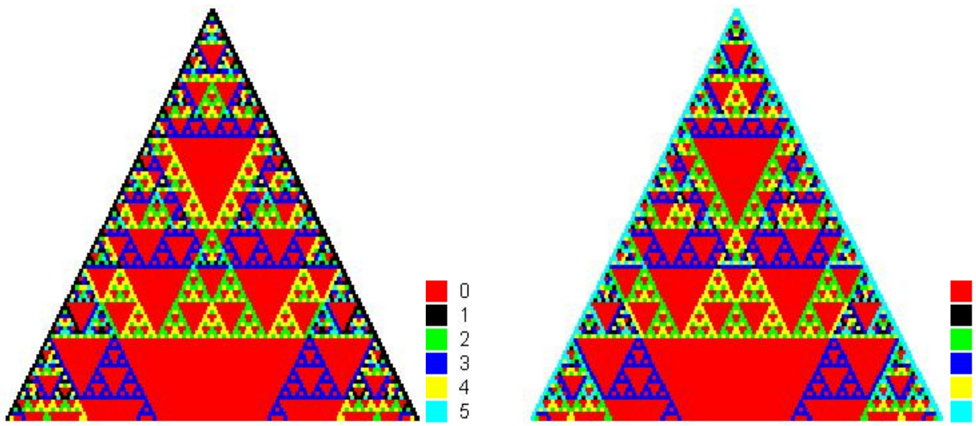
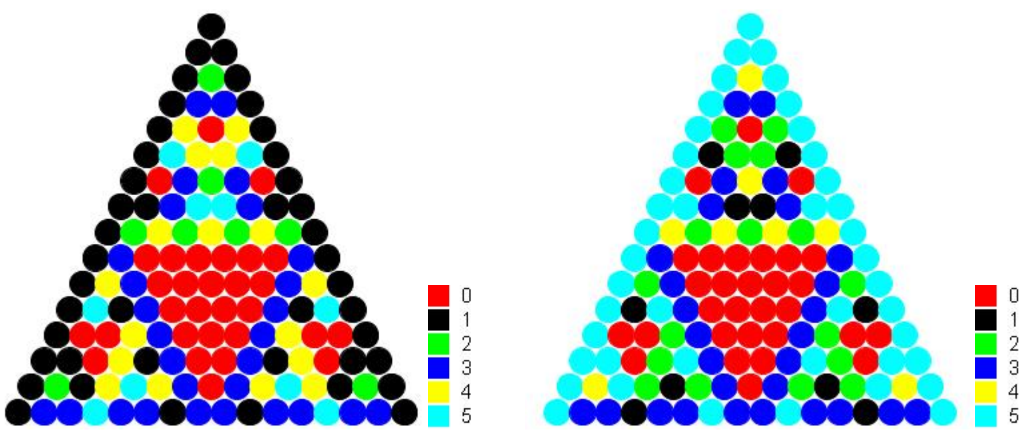
Isomorphism

Several of the PascGalois labs deal with the concept of isomorphism in some way, mainly because the PascGalois JE program produces images that can be considered signatures of the group structure. So determining if two groups are isomorphic is sometimes as simple as seeing if the two images look the same. In some cases signatures of different groups can look very close to each other but those for $\mathbb{Z}_n$ do tend to be noticeably distinct for relatively small n.

The PascGalois JE program has another feature that allows the user to visualize isomorphism. The colors in the color correspondence box on the right can be dragged and dropped into any other color correspondence box. So if you have two windows open that have what you feel are isomorphic groups you can test your isomorphism by "dragging" out the isomorphism and then refreshing the image. If the result is *exactly* the same image then you have an isomorphism. Here is an easy example, it should be no surprise that $\mathbb{Z}_6 \cong \mathbb{Z}_6$. But what if we created one triangle using a single 1 as the seed and the other using a single 5 as the seed? Well we would get the following two images.



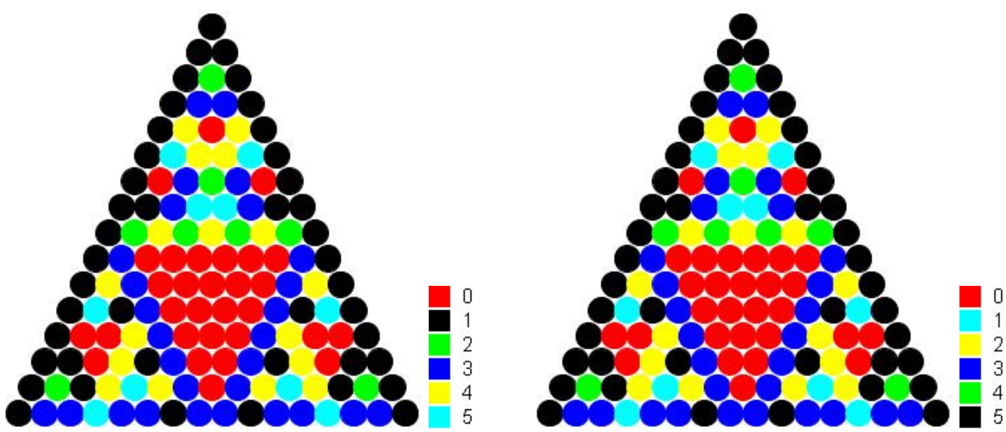
These obviously have the same structure but their colorings are not exactly the same. This is precisely the concept of isomorphism, the structure is the same it is just that the elements are renamed, or in this case reorganized. Since an isomorphism is a map of elements from one group to another we need to be able to see the the color changes between the two images above. So we should zoom in a bit. Now we could zoom in on any particular area of both triangles but it would be easiest to zoom in on the top.



Looking at the very top element we see that black gets sent to light blue, so 1 gets mapped to 5. This should not be too much of a surprise since we used 1 to generate the first triangle and 5 to generate the second. From the $(2, 1)$ position we see that green goes to yellow, so 2 gets mapped to 4. From the $(3, 1)$ position we see that blue goes to blue, so 3 gets mapped to 3. From the $(4, 1)$ position we see that yellow goes to green, so 4 gets mapped to 2. From the $(5, 1)$ position we see that light blue goes to black, so 5 gets mapped to 1. Finally, from the $(4, 2)$ position we see that red goes to red, so 0 gets mapped to 0. So our proposed isomorphism $\phi : \mathbb{Z}_6 \to \mathbb{Z}_6$ is

$$
\begin{aligned}
\phi(0) &= 0 \\
\phi(1) &= 5 \\
\phi(2) &= 4 \\
\phi(3) &= 3 \\
\phi(4) &= 2 \\
\phi(5) &= 1
\end{aligned}
$$

We say proposed isomorphism because seeing where just a couple elements go is not enough. As you know from class an isomorphism must be a one-to-one correspondence that preserves the group operation, that is, $\phi(a * b) = \phi(a) *' \phi(b)$. We clearly have a one-to-one correspondence but how can we check the preservation of the operation? We could of course do this by hand but visually an isomorphism would mean that once we "renamed" the elements the two group structures are *exactly* the same. So if we were to recolor the second triangle using this proposed isomorphism and then refresh the second triangle we should see *exactly* the same image. To drag out the isomorphism place the mouse over the element 1 in the first triangle's color correspondence click and drag it over to the element 5 in the second triangle's color correspondence and then release the mouse button. At this point the element 5 in the second triangle's color correspondence should be the color black. We could refresh the image but lets wait until we are finished with the proposed isomorphism. Now click and drag the 2 to 4, then 4 to 2 and finally the 5 to 1. Now refresh the image. The images should look like the following.



Since the two triangles are exactly the same we know that the operation is preserved and hence we do have an isomorphism. With an example this easy it could be just as quick to have done the calculations by hand but with more complicated structures this method can be less work. Furthermore, it is easy to spot when you have a map that does not preserve the group operation because some parts the triangles will match and some parts will not.

In this lab we will investigate the groups $\mathbb{Z}_n \times \mathbb{Z}_m$. In particular, we ask the question, "When is the group $\mathbb{Z}_n \times \mathbb{Z}_m$ cyclic?"

Recall that the group $\mathbb{Z}_n \times \mathbb{Z}_m$, which we should write as $\langle \mathbb{Z}_n \times \mathbb{Z}_m, \oplus \rangle$ is the set of all ordered pairs (a, b) where $a \in \mathbb{Z}_n$ and $b \in \mathbb{Z}_m$. The operation $\oplus$ is defined as $(a, b) \oplus (c, d) = (a + c, b + d)$, where the first component is computed mod n and the second component is computed mod m.

Before we get into the questions we need some preliminary results. These results will cut down on the amount of work we will need to do. If you have not proved these results in class you should do so now. First of all, we know that we can generate $\mathbb{Z}_n \times \mathbb{Z}_m$ with the two elements $(1, 0)$ and $(0, 1)$. So we can generate the group using two generators. But if are asking the question of when this group is cyclic then we want to know when it can be generated by a single element. Obviously, neither of the elements $(1, 0)$ or $(0, 1)$ will do the trick since one of the components is 0 and hence we will never generate any other element in that component. From is observation is it clear that to even have a chance of generating all of $\mathbb{Z}_n \times \mathbb{Z}_m$, we need an element of the form (a, b) where a is a generator of $\mathbb{Z}_n$ and b is a generator of $\mathbb{Z}_m$. We should be careful here since this criterion alone may still not be enough but it at least narrows down the possibilities.

Exercises:

1. This series of questions will concentrate on $\mathbb{Z}_2 \times \mathbb{Z}_3$
   (a) What are the generators of $\mathbb{Z}_2$?
   (b) What are the generators of $\mathbb{Z}_3$?
   (c) How many elements are in $\mathbb{Z}_2 \times \mathbb{Z}_3$?
   (d) Create all ordered pairs (a, b) where a is a generator of $\mathbb{Z}_2$ and b is a generator of $\mathbb{Z}_3$.
   (e) Graph the triangle for $\mathbb{Z}_2 \times \mathbb{Z}_3$ generated by each of the pairs from the previous exercise.
   If you have not worked with the groups $\mathbb{Z}_n \times \mathbb{Z}_m$ in the PascGalois JE program follow this procedure. Select the Group tab, then for the group type select `Zn X Zm (Addition)`, and type in 2 into the n box and 3 into the m box. Click on the 1-D Automaton tab and in the Default Element box of the Options tab you should see $(0, 0)$. If you are using an older version of the PascGalois JE program the default element may not have changed, in this case click the Use Group Identity button and you should now see $(0, 0)$ in the Default Element box. Click on the Seed tab and in the grid input one of the ordered pairs created in part (1d), for example $(1, 1)$. The PascGalois JE program uses the same notation for elements of $\mathbb{Z}_n \times \mathbb{Z}_m$ as we would, a comma separated ordered pair in parentheses. Now click on the Image tab and graph the image.
   (f) How many elements of $\mathbb{Z}_2 \times \mathbb{Z}_3$ are in the triangle you produced?
   (g) Does this show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic? Why?
   (h) Does the image look like any previous triangle you have graphed? If so, which one? Can you construct an isomorphism between the two structures? Give the isomorphism in terms of elements as well as colors.
   (i) Repeat the process with another ordered pair from part (1d).

2. For each of the following groups, create all ordered pairs (a, b) where a is a generator of $\mathbb{Z}_n$ and b is a generator of $\mathbb{Z}_m$. Graph each of the triangles generated by these ordered pairs and determine which of the following groups are cyclic.
   (a) $\mathbb{Z}_2 \times \mathbb{Z}_4$
   (b) $\mathbb{Z}_2 \times \mathbb{Z}_5$
   (c) $\mathbb{Z}_3 \times \mathbb{Z}_4$
   (d) $\mathbb{Z}_2 \times \mathbb{Z}_2$
   (e) $\mathbb{Z}_4 \times \mathbb{Z}_6$
   (f) $\mathbb{Z}_3 \times \mathbb{Z}_6$
   (g) $\mathbb{Z}_3 \times \mathbb{Z}_5$
   (h) $\mathbb{Z}_{10} \times \mathbb{Z}_{21}$

3. From all of the examples you have done above, make a conjecture about the relationship between n and m which makes $\mathbb{Z}_n \times \mathbb{Z}_m$ cyclic. Write your conjecture in the form

   If ______________, then $\mathbb{Z}_n \times \mathbb{Z}_m$ is a cyclic group, moreover $\mathbb{Z}_n \times \mathbb{Z}_m \cong$ ________.

4. As we mentioned above the groups $\mathbb{Z}_n \times \mathbb{Z}_m$ can always be generated using two generators, specifically, $(1, 0)$ and $(0, 1)$. In each of the above exercises that were not cyclic regraph them using these two ordered pairs as generators. You will have to increase the number of columns in the seed to two and put in $(1, 0)$ into the first cell and $(0, 1)$ into the second cell. Graph each of the $\mathbb{Z}_n \times \mathbb{Z}_m$ beside the image of the group they would be isomorphic to if they were cyclic, what are the similarities and differences in the triangular images?